

Sensitivity of LATE Estimates to Violations of the Monotonicity Assumption

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Abstract

In this paper, we develop a method to assess the sensitivity of local average treatment effect estimates to potential violations of the monotonicity assumption of Imbens and Angrist (1994). We parameterize the degree to which monotonicity is violated using two sensitivity parameters: the first one determines the share of defiers in the population, and the second one measures differences in the distributions of outcomes between compliers and defiers. For each pair of values of these sensitivity parameters, we derive sharp bounds on the outcome distributions of compliers in the first-order stochastic dominance sense. We identify the robust region that is the set of all values of sensitivity parameters for which a given empirical conclusion, e.g. that the local average treatment effect is positive, is valid. Researchers can assess the credibility of their conclusion by evaluating whether all the plausible sensitivity parameters lie in the robust region. We obtain confidence sets for the robust region through a bootstrap procedure and illustrate the sensitivity analysis in an empirical application. We also extend this framework to analyze treatment effects of the entire population.

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1. INTRODUCTION

The local average treatment effect framework (LATE) is used for instrumental variable analysis in setups of heterogeneous treatment effects (Imbens and Angrist, 1994). We consider settings of a binary instrumental variable and a binary treatment variable. The Wald estimand then equals the treatment effect of *compliers*, individuals for which the instrument influences the treatment status, given the well-known classical LATE assumptions: monotonicity, independence, and relevance. Monotonicity states that the effect of the instrument on the treatment decision is monotone across all units. In the canonical example, in which the instrument encourages units to take up the treatment, monotonicity rules out the existence of *defiers*, i.e., units that receive the treatment only if the instrument discourages them. Researchers might question the validity of this assumption in empirical applications. In these settings, the local treatment effect estimates might be biased and might lead the researchers to draw incorrect conclusions about the true treatment effect.

As an example of a setup in which monotonicity could plausibly be violated, consider the study of Angrist and Evans (1998), who analyze the effect of having a third child on the labor market outcomes of mothers. As the decision to have a third child is endogenous, the authors use a dummy for whether the first two children are of the same sex as an instrument. The underlying reasoning is that some parents would only decide to have a third child if their first two children were of the same sex; these parents are compliers. The monotonicity assumption seems questionable in this setting as parents, who have a strong preference for one specific sex, might act as a defier in this setup. Consider, for example parents who want to have at least two boys and their first child is a boy. Contrary to the incentive given by instrument, they have two children if their second child is a boy, and three children if their second child is a girl. As the monotonicity assumption might be questionable in this example, one can question the validity of empirical conclusions drawn from the classical LATE analysis.¹

In this paper, we provide a framework to evaluate the sensitivity of treatment effect estimates to a potential violation of the monotonicity assumption. As noted in Angrist, Imbens, and Rubin (1996), a violation of the monotonicity assumption always has two dimensions: The first dimension is the heterogeneous effect of the instrumental variable on the treatment variable, the presence of defiers. The second dimension is the heterogeneous effect of the treatment variable on the outcome variable, the outcome heterogeneity between defiers and compliers. We derive the degree to which monotonicity is violated by parameterizing these two dimensions.

¹The other LATE assumptions seem to be plausible here. As the sex of a child is determined by nature and as only the number of and not the sex of the child arguably influences the labor market outcome of mothers, the independence assumption seems to be satisfied. The relevance assumption is testable.

We parameterize the existence of defiers by their population size and the outcome heterogeneity by the Kolmogorov-Smirnov norm, which bounds the difference of the cumulative distribution functions of compliers and defiers. For each of these two sensitivity parameters, we identify sharp bounds of the outcome distribution of compliers in a first-order stochastic dominance sense. These bounds also imply sharp bounds on various treatment effects, e.g., the average treatment effect or quantile treatment effects of compliers.

Our analysis proceeds in two steps. In a first step, we identify the *sensitivity region*. The sensitivity region defines the set of sensitivity parameters for which a data generating process exists, that is consistent with our model assumptions and implies both the observed probabilities and the sensitivity parameters. Since sensitivity parameters lying in the complement of the sensitivity region are not compatible with our model, we do not analyze them further. For the derivation of the sensitivity region, we also derive sharp bounds of the population size of defiers. In a second step, we identify the *robust region*, which is the set of sensitivity parameters that imply treatment effects that are consistent with a particular empirical conclusion; for instance, the treatment effect of compliers has a specific sign or a particular order of magnitude.² Parameters lying in the complement of the robust region, the *nonrobust region*, imply treatment effects that are not, or may not be, consistent with the given empirical conclusion. The robust region and the nonrobust region are separated from each other by the *breakdown frontier*, following the terminology of Masten and Poirier (2020). For each population size of defiers, the breakdown frontier identifies the weakest assumption about outcome heterogeneity, which is necessary to be imposed to imply treatment effects being consistent with the particular empirical conclusion under consideration.

This framework can be used in the following ways. First, by evaluating the size of the sensitivity region, one can determine the plausibility of the model. If this set is empty, the model is refuted, which implies that even if one would allow for an arbitrary violation of the monotonicity assumption, at least one of the model assumptions has to be violated. Second, researchers can analyze the sensitivity of their estimates with respect to the degree to which the monotonicity assumption is violated by varying the sensitivity parameters within the sensitivity region. Third, by evaluating the plausibility of the parameters within the robust region, researchers can assess the sign or the order of magnitude of the treatment effect. While being transparent about the imposed assumptions, they might still arrive at a particular empirical conclusion of interest in a credible way. Fourth, one can assess to which degree monotonicity has to be violated to overturn a particular empirical conclusion. Within our framework, researchers can use their economic insights about the analyzed situation to judge the severity of a violation monotonicity.

²See Masten and Poirier (2020) for a detailed exposition of this approach.

While the main focus of this paper lies on treatment effects of compliers, we also show how this framework can be exploited to analyze treatment effects of the entire population. Under further support assumptions of the outcome variable and for given sensitivity parameters, the average treatment effect of the entire population is partially identified, which complements known results in the literature (see Balke and Pearl, 1997; Kamat, 2018; Kitagawa, 2021; Machado, Shaikh, and Vytlačil, 2019). Since the analytic expressions of the sensitivity and robust regions are rather complicated and difficult to interpret, we provide simplified analytical expressions of these regions for a binary outcome.

To construct confidence sets for both the sensitivity and the robust region, we show that both regions are determined through mappings of some underlying parameters. These mappings are not Hadamard-differentiable, and inference methods relying on standard Delta method arguments are therefore not applicable. We show how to construct smooth mappings that bound the parameters of interest. This construction leads to mappings for which standard Delta method arguments are applicable, and we use the nonparametric bootstrap to construct valid confidence sets for the parameters of interest. With a binary outcome variable, the mappings resulting in the sensitivity and robust region are considerably simpler. Therefore, we can use a generalized Delta method to show asymptotic distributional results and apply a bootstrap procedure to construct asymptotically valid confidence sets.

We show in a Monte Carlo study that our proposed inference method has good finite sample properties. We further apply our method to the setup studied by Angrist and Evans (1998) introduced above. We show that relatively strong assumptions on either the population size of the defiers or the outcome heterogeneity have to be imposed to preserve the sign of the estimated treatment effect. This result demonstrates that the monotonicity assumption is key in the local treatment effect framework.

The remainder of this paper is structured as follows: A literature review follows, and Section 2 illustrates the setup in a simplified setting. Section 3 introduces the sensitivity parameters and Section 4 derives sharp bounds on the distribution functions of compliers. The main sensitivity analysis is presented in Section 5. Section 6 discusses extensions and Section 7 derives estimation and inference results. Section 8 contains a simulation study and Section 9 an empirical example. Section 10 concludes. All proofs and additional materials are deferred to the appendix.

Literature. This paper relates to several strands of the literature. First, this paper contributes to the growing strand of the literature, which considers sensitivity analysis in various applications. These applications include, among many others, violations of parametric assumptions, violations of moment conditions, and multiple examples within the treatment effect literature (see, among

others, Andrews, Gentzkow, and Shapiro, 2017, 2020a,b; Andrews and Shapiro, 2020; Armstrong and Kolesár, 2021; Bonhomme and Weidner, 2018, 2019; Chen, Tamer, and Torgovitsky, 2011; Christensen and Connault, 2019; Conley, Hansen, and Rossi, 2012; Imbens, 2003; Kitamura, Otsu, and Evdokimov, 2013; Mukhin, 2018; Roth and Rambachan, 2019). This paper is closely related to the literature about breakdown points of Horowitz and Manski (1995); Imbens (2003); Kline and Santos (2013); Stoye (2005, 2010), and especially closely related to Masten and Poirier (2020, 2021). These papers consider several assumptions in the treatment effect literature, but not the monotonicity assumption.

Second, it is related to the local average treatment effect framework literature, which is formally introduced in Imbens and Angrist (1994) and further in Vytlacil (2002). Several papers consider violations of the monotonicity assumption through different types of assumptions. Balke and Pearl (1997); Huber (2015); Huber, Laffers, and Mellace (2017); Huber and Mellace (2015); Machado et al. (2019); Manski (1990) consider a binary and Kitagawa (2021) a continuous outcome variable and partially identify the average treatment effect. Dahl, Huber, and Mellace (2017); Huber et al. (2017); Manski and Pepper (2000); Small, Tan, Ramsahai, Lorch, Brookhart et al. (2017) propose alternative assumptions on the data generating process, which are strictly weaker than monotonicity and obtain bounds on various treatment effects. Richardson and Robins (2010) consider a binary outcome variable and derive bounds on outcome distributions for a given population size of always takers. We consider not only a binary outcome variable and we introduce a second parameter to also bound outcome heterogeneity. We then consider these sensitivity parameters within the framework of a breakdown frontier.

De Chaisemartin (2017) shows that in the presence of defiers, under certain assumptions, the Wald estimand still identifies a convex combination of causal treatment effects of only a subpopulation of compliers. In a policy context, the treatment effect of compliers might be of particular interest because the treatment status of compliers is most likely to change with a small policy change. However, the same reasoning does not apply to the subpopulation of compliers. Klein (2010) evaluates the sensitivity of the treatment effect of compliers to random departures from monotonicity. Fiorini, Stevens et al. (2014) give examples of analyzing the sensitivity of the monotonicity, and Huber (2014) considers a violation of monotonicity in a specific example. They do not provide sharp identification results of the treatment effect of compliers in the presence of defiers, nor do they derive the robust region. A violation of the monotonicity assumption with a non-binary instrumental variable is considered, and alternative assumptions and testing procedures are proposed in Frandsen, Lefgren, and Leslie (2019); Mogstad, Torgovitsky, and Walters (2019); Norris, Pecenco, and Weaver (2020). This paper contributes to this literature by presenting an

effective tool to analyze the severity of a potential violation of the monotonicity assumption. It thus gives applied researchers a new tool to evaluate the robustness of their estimates to a violation of the monotonicity assumption, and their estimates may thereby gain credibility.

Our proposed inference procedure builds on seminal work about Delta methods for non-differentiable mappings by Dümbgen (1993); Fang and Santos (2018); Hong and Li (2018); Shapiro (1991), and it further exploits ideas of smoothing population parameters by Chernozhukov, Fernández-Val, and Galichon (2010); Haile and Tamer (2003); Masten and Poirier (2020).

2. SETUP

2.1. Model of the Local Average Treatment Effect. We observe the distribution of the random variables (Y, D, Z) , where Y is the outcome of interest; D is the actual treatment status, with $D = 1$ if the person is treated and $D = 0$ otherwise; and Z is the instrument, with $Z = 1$ if the person is assigned to treatment and $Z = 0$ otherwise. We assume that each unit has potential outcomes Y_0 in the absence and Y_1 in the presence of treatment, and potential treatment status D_1 when assigned to treatment and D_0 when not assigned to treatment. The observed and potential outcomes are related by $Y = DY_1 + (1 - D)Y_0$, and observed and potential treatment status by $D = ZD_1 + (1 - Z)D_0$.

Based on the effect of the instrument on the treatment status, we distinguish four different groups: compliers that are only treated if they are assigned to treatment (CO); defiers that are only treated if they are not assigned to treatment (DF); always takers that are independently of the instrument always treated (AT), and never takers that are never treated (NT). We denote the population sizes of the respective group by π_{AT} , π_{NT} , π_{CO} , and π_{DF} . We denote by Y_d^T the potential outcome variable of group $T \in \{AT, NT, CO, DF\}$ under treatment status d . To simplify the notation, we write Y_d^{dT} for the potential outcome variable of always takers if $d = 1$ and otherwise of never takers, and similarly π_{dT} for the respective population size. We denote the outcome distribution of a variable Y by F_Y , its density function, if it exists, by f_Y , and its support by \mathbb{Y} .³

The key parameters of interest in this analysis are treatment effects of compliers. We denote the average treatment effect of compliers by⁴

$$\Delta_{CO} = \mathbb{E}[Y_1 - Y_0 | D_0 = 0, D_1 = 1].$$

Throughout the paper, we assume that $\mathbb{P}(D = 1 | Z = 1) \geq \mathbb{P}(D = 1 | Z = 0)$ without loss of

³Throughout the paper, we implicitly assume that all necessary moments of all random variables for the parameter of interest exist; for instance, if we consider the local average treatment effect, we assume Y_d^T has first moments for all $d \in \{0, 1\}$ and $T \in \{C, DF, AT, NT\}$.

⁴Similarly, the average treatment effect of defiers is denoted by $\Delta_{DF} = \mathbb{E}[Y_1 - Y_0 | D_0 = 1, D_1 = 0]$.

generality, and we impose the following identifying assumptions.

Assumption 1. *The instrument satisfies $(Y_1, Y_0, D_1, D_0) \perp Z$ (Independence), and $\mathbb{P}(D = 1|Z = 1) > \mathbb{P}(D = 1|Z = 0)$ (Relevance).*

We refer to Angrist et al. (1996) for an extensive discussion of these assumptions.

2.2. Illustration of the Sensitivity Analysis. In this section, we illustrate the sensitivity analysis in a very simplified framework, where we introduce the sensitivity parameters, the sensitivity and the robust region. In contrast to our main sensitivity analysis in Section 3-5, we do not consider any sharp identification results in this illustration.

2.2.1. Sensitivity Parameter Space. In the presence of defiers, Angrist et al. (1996) show that the average treatment effect of compliers is not point identified. The Wald estimand, $\beta^{IV} = \text{Cov}(Y, Z)/\text{Cov}(D, Z)$, equals a weighted difference of the average treatment effect of compliers and defiers:

$$\beta^{IV} = \frac{1}{\pi_{CO} - \pi_{DF}} (\pi_{CO}\Delta_{CO} - \pi_{DF}\Delta_{DF}). \quad (1)$$

Three parameters in equation (1) are in general not identified: the population size of defiers π_{DF} , the treatment effect of compliers Δ_{CO} and of defiers Δ_{DF} .⁵ To bound the average treatment effect of compliers, we introduce two sensitivity parameters. The first one determines the population size of defiers, and the second one outcome heterogeneity between compliers and defiers. These two parameters measure the degree to which monotonicity is violated and represent the two dimensions of heterogeneity: (i) heterogeneous effects of the instrument on the treatment status and (ii) heterogeneous effects of the treatment on the outcome.

The heterogeneous impact of the instrument on the treatment status, is parameterized, in the most simplest ways, by the population size of defiers

$$\pi_{DF} = \mathbb{P}(D_0 = 1 \text{ and } D_1 = 0). \quad (2)$$

A larger sensitivity parameter π_{DF} implies a more severe violation of monotonicity. It is clear that, for a given population size of defiers, π_{DF} , the population sizes of the other groups are point identified. In our analysis, these population sizes are, therefore, functions of the sensitivity parameter π_{DF} , but we leave this dependence implicit.⁶

⁵Clearly, if either $\pi_{DF} = 0$, implying the absence of defiers, or $\Delta_{CO} = \Delta_{DF}$, implying that compliers and defiers have the same average treatment effect, the treatment effect Δ_{CO} is still point identified.

⁶It follows from the definitions of the groups and our assumptions that $\pi_{AT} = \mathbb{P}(D = 1|Z = 0) - \pi_{DF}$, $\pi_{NT} = \mathbb{P}(D = 0|Z = 1) - \pi_{DF}$ and $\pi_{CO} = \mathbb{P}(D = 1|Z = 1) - \mathbb{P}(D = 1|Z = 0) + \pi_{DF}$.

We parameterize the second dimension of heterogeneity by the sensitivity parameter δ_a which equals the absolute differences in treatment effects of both groups

$$\delta_a = |\Delta_{CO} - \Delta_{DF}|.$$

A larger sensitivity parameter δ_a implies a more severe violation of monotonicity.

2.2.2. Sensitivity Region and Robust Region. The *sensitivity region* is the set of sensitivity parameters which do not violate our model assumptions. For instance, a sensitivity parameter $\pi_{DF} \geq 0.5$ would violate our model assumptions as the relevance assumption implies that $\pi_{CO} > \pi_{DF}$. Therefore, such a sensitivity parameter does not lie within our sensitivity region, which is identified without imposing any additional assumptions. In this illustrative example, we simplify the derivation and say that the sensitivity region is trivially given by

$$SR_a = [0, 0.5) \times \mathbb{R}_+.$$

In our main sensitivity analysis, this set, however, is nontrivial and can be empty. In this case, the model is rejected, implying that even though the monotonicity assumption may be violated, at least one of the other model assumptions has to be violated as well.

Even though the treatment effect of compliers is generally not point identified if $\pi_{DF} > 0$, using (1), it is partially identified for any given pair of sensitivity parameters (π_{DF}, δ_a) by

$$\Delta_{CO} \in \left[\beta^{IV} - \frac{\pi_{DF}}{\pi_{CO} - \pi_{DF}} \delta_a, \beta^{IV} + \frac{\pi_{DF}}{\pi_{CO} - \pi_{DF}} \delta_a \right].$$

In a typical sensitivity analysis, researchers now consider different values of the sensitivity parameters to evaluate the identified sets of the parameter of interest and to evaluate the robustness of the LATE estimates to a potential violation of monotonicity. However, in many empirical applications, the interest does not lie in the precise treatment effect but in its sign or in its order of magnitude. It is, therefore, natural to start with the empirical conclusion of interest and to ask which sensitivity parameters imply treatment effects that are consistent with this conclusion. This approach is formalized by the breakdown frontier (see, e.g., Kline and Santos, 2013; Masten and Poirier, 2020).

We now consider the empirical conclusion that $\Delta_{CO} \geq \mu$, and we assume that $\beta^{IV} \geq \mu$.⁷ Under our model assumptions and for a given value of the population size of defiers π_{DF} , the breakdown point determines the largest value of outcome heterogeneity δ_a that implies treatment effects that are consistent with our empirical conclusion of interest. Specifically, for any $\pi_{DF} \in [0, 0.5]$, the

⁷If $\beta^{IV} \leq \mu$, the robust region for the conclusion that $\Delta_{CO} \geq \mu$ is empty.

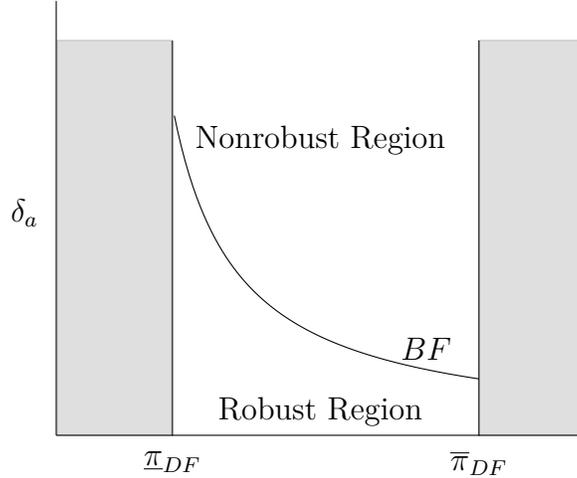


Figure 1: Illustration of Sensitivity and Robust Region. Non-shaded area represents sensitivity region. $[\underline{\pi}_{DF}, \bar{\pi}_{DF}]$ represent some bounds on the population size of defiers.

breakdown point is given by

$$BP_a(\pi_{DF}) = \frac{\pi_{CO} - \pi_{DF}}{\pi_{DF}}(\beta^{IV} - \mu).$$

The breakdown frontier (BF) is the set of all breakdown points and the robust region (RR) is the set of all sensitivity parameters that are consistent with the empirical conclusion of interest. They are respectively given by

$$BF_a = \{(\pi_{DF}, BP_a(\pi_{DF})) \in SR_a\} \quad \text{and} \quad RR_a = \{(\pi_{DF}, \delta_a) \in SR_a : \delta_a \leq BP_a(\pi_{DF})\}.$$

The nonrobust region is the complement of the robust region within the sensitivity region. It contains sensitivity parameters that may or may not be consistent with the empirical conclusion. Due to the functional form of the breakdown frontier, the nonrobust region is a convex set in this example. An illustrative example of this setup is shown in Figure 1.

In this simple example, neither the sensitivity region nor the robust regions are sharp. For example, if the outcome is binary than the difference between compliers and defiers treatment effects is clearly bounded. Similarly, the robust region might also be substantially reduced by taking into account the actually observed outcomes.⁸ This reasoning means that even though a parameter pair may lie within the sensitivity region, it might not imply a well-defined data generating process that is consistent with the model assumptions and the observed probabilities. Similarly, even though

⁸To give a concrete example, assume that all treated units have a realized outcome of 1 and all nontreated units have a realized outcome of 0. Then it is clear, that the treatment effect of compliers is point identified to be one.

a parameter pair may lie within the nonrobust region, it might be robust. Empirical conclusions that can be drawn from this analysis might, therefore, not be very informative. Consequently, we improve upon this framework in the remainder of this paper.

3. SENSITIVITY PARAMETERS

In this section, we introduce two sensitivity parameters that are interpretable and imply bounds on the outcome distributions of compliers so that the parameter of interest is partially identified. They allow us to consider a trade-off between the strength of the imposed assumption and the size of the identified set. To derive the sensitivity parameters, we consider the following function:

$$G_d(y) = \frac{\text{Cov}(\mathbf{1}\{Y \leq y\}, \mathbf{1}\{D = d\})}{\text{Cov}(Z, \mathbf{1}\{D = d\})},$$

for $d \in \{0, 1\}$. In the absence of defiers, $G_d(y)$ is the cumulative distribution function of compliers under treatment status d . In the presence of defiers, it holds analogously to (1) that

$$G_d(y) = \frac{1}{\pi_{\text{CO}} - \pi_{\text{DF}}} \left(\pi_{\text{CO}} F_{Y_d^{\text{CO}}}(y) - \pi_{\text{DF}} F_{Y_d^{\text{DF}}}(y) \right). \quad (3)$$

The outcome distributions of compliers are thus identified up to the population size of defiers and the heterogeneity between the outcome distributions of compliers and defiers. We introduce two sensitivity parameters to parameterize these two dimensions. First, the presence of defiers is parameterized by the population size of defiers π_{DF} (2). Second, outcome heterogeneity is represented by δ , which bounds the maximal difference between cumulative distribution functions of the outcome of compliers and defiers by the Kolmogorov-Smirnov (KS) norm

$$\max_{d \in \{0, 1\}} \sup_{y \in \mathbb{Y}} \{|F_{Y_d^{\text{CO}}}(y) - F_{Y_d^{\text{DF}}}(y)|\} = \delta,$$

where $\delta \in [0, 1]$. Without a restriction on δ , the outcome distributions can be arbitrarily different. If $\delta = 0$, the outcome distributions are restricted the most as both distribution functions coincide. A larger value of the parameter δ implies a more severe violation of monotonicity.

There are clearly many different possibilities for how heterogeneity between distribution functions can be specified. In this paper, we choose the Kolmogorov-Smirnov norm, as it leads to tractable analytical solutions of the bounds on the compliers outcome distribution. More importantly, this parameterization is simple enough to be interpretable in an empirical conclusion.

A similar parameterization is chosen in Kline and Santos (2013) in a different context.⁹

⁹Since the parameterization of δ is weak on the tails of the distributions, the bounds on the tails are likely to be

4. PARTIAL IDENTIFICATION OF DISTRIBUTION FUNCTIONS

Since our main sensitivity analysis exploits bounds on parameters defined by the distribution function $F_{Y_d^{CO}}$ for $d \in \{0, 1\}$, we bound this distribution function for a fixed given sensitivity parameter pair (π_{DF}, δ) in this section. We illustrate the derivation of the bounds in the subsequent sections, and the main result is stated in Section 4.3.

4.1. Preliminaries.

4.1.1. *Identification Strategy.* Our goal is to obtain sharp lower and upper bounds of the distribution function $F_{Y_d^{CO}}$ in a first-order stochastic dominance sense. That is, we derive analytical characterizations of the distribution functions $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$ that are feasible candidates for $F_{Y_d^{CO}}$, in the sense that they are compatible with the imposed sensitivity parameters, our assumptions, and the population distributions of observable probabilities. They are further such that $\underline{F}_{Y_d^{CO}}(y) \leq F_{Y_d^{CO}}(y) \leq \overline{F}_{Y_d^{CO}}(y)$, for all $y \in \mathbb{Y}$. The identification strategy for deriving such sharp bounds $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$ is based on the premise that any candidate distribution function of $F_{Y_d^{CO}}$ then also implies distribution functions of $F_{Y_d^{dT}}$ and $F_{Y_d^{DF}}$. Our candidate function $F_{Y_d^{CO}}$ is therefore feasible, only if the implied functions of $F_{Y_d^{dT}}$ and $F_{Y_d^{DF}}$ are indeed distribution functions.

The explicit analytical characterization of these sharp bounds illustrates the effect of the sensitivity parameters on the bounds, and more importantly, it implies sharp bounds on a variety of treatment effects of interest, e.g., the average treatment effect of compliers (Stoye, 2010, Lemma 1).¹⁰

4.1.2. *Notation.* We here collect the notation used in the following subsections. Let $d, s \in \{0, 1\}$ and $y \in \mathbb{Y}$. Let the differences in population sizes of compliers and defiers be denoted by $\pi_{\Delta} = \pi_{CO} - \pi_{DF}$. Let $Q_{ds}(y) \equiv \mathbb{P}(Y \leq y, D = d | Z = s)$ be the observed joint distribution of Y and D . We further let, for \mathcal{B} denoting the Borel σ -algebra,

$$\tilde{G}_d^+(y) = \sup_{B \in \mathcal{B}} \{ \mathbb{P}(Y \in B, Y \leq y, D = d | Z = d) - \mathbb{P}(Y \in B, Y \leq y, D = d | Z = 1 - d) \}.$$

and $G_d^+ = \frac{1}{\pi_{CO}} \tilde{G}_d^+(y)$. Our sensitivity analysis is based on the following observed underlying parameters

$$\theta = (Q_{11}, Q_{10}, Q_{01}, Q_{00}, \tilde{G}_1^+, \tilde{G}_0^+). \tag{4}$$

uninformative. Imposing a *weighted* KS assumption, that penalizes deviations at the tails of the two distributions more, would overcome this issue but would also lead to less tractable results.

¹⁰The explicit characterization also allows the inference procedure to be based on $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$.

4.2. Preliminary Bounds. To illustrate the identification argument, we first derive preliminaries bounds on the distribution function $F_{Y_d^{CO}}$, which are not necessarily sharp in general. Based on the law of total probability and our assumptions, the probability function Q_{dd} is a weighted average of the distribution functions $F_{Y_d^{CO}}$ and $F_{Y_d^{dT}}$, specifically $Q_{dd}(y) = \pi_{CO}F_{Y_d^{CO}}(y) + \pi_d F_{Y_d^{dT}}(y)$. Any feasible distribution function of $F_{Y_d^{CO}}$ has to imply a function $F_{Y_d^{dT}}$ that is a distribution function. Exploiting this argument and using our sensitivity parameter π_{DF} , it follows that

$$\frac{1}{\pi_{CO}}Q_{dd}(y) \leq F_{Y_d^{CO}}(y) \leq \frac{1}{\pi_{CO}}(Q_{dd}(y) - \pi_d). \quad (5)$$

These bounds correspond to the extreme scenarios where compliers have the highest or the lowest outcomes compared to always and never takers.

Using the same argument for defiers and the definition of $G_d(y)$ in (3), it further follows that

$$\frac{\pi_{\Delta}}{\pi_{CO}}G_d(y) \leq F_{Y_d^{CO}}(y) \leq \frac{1}{\pi_{CO}}(\pi_{\Delta}G_d(y) + \pi_{DF}). \quad (6)$$

We now consider the second sensitivity parameter δ . Based on the definition of $G_d(y)$ in (3), we conclude that any feasible candidate of $F_{Y_d^{CO}}$ also has to satisfy that

$$G_d(y) - \frac{\pi_{DF}}{\pi_{\Delta}}\delta \leq F_{Y_d^{CO}}(y) \leq G_d(y) + \frac{\pi_{DF}}{\pi_{\Delta}}\delta. \quad (7)$$

Since the function G_d is not necessarily increasing in y for all $y \in \mathbb{Y}$, bounds on the distribution function $F_{Y_d^{CO}}$ based on (6) and (7) have to take this into account. We therefore directly consider bounds on $F_{Y_d^{CO}}$ that employ this information. To be precise, for the lower bound, we consider equation (6) and (7), where we replace G_d by its smallest, nondecreasing upper envelope; vice versa, for the upper bound, where we replace G_d by its greatest, nondecreasing lower envelope.¹¹ Following this reasoning and taking (5)-(7) into account, the lower bound is given by

$$\underline{H}_{Y_d^{CO}}(y, \pi_{DF}, \delta) = \max\left\{0, \frac{1}{\pi_{CO}}(Q_{dd}(y) - \pi_d), \frac{\pi_{\Delta}}{\pi_{CO}} \sup_{\tilde{y} \leq y} G_d(\tilde{y}), \sup_{\tilde{y} \leq y} G_d(\tilde{y}) - \frac{\pi_{DF}}{\pi_{\Delta}}\delta\right\}, \quad (8)$$

and the upper bound by

$$\overline{H}_{Y_d^{CO}}(y, \pi_{DF}, \delta) = \min\left\{1, \frac{1}{\pi_{CO}}Q_{dd}(y), \frac{\pi_{\Delta}}{\pi_{CO}}(\inf_{\tilde{y} \geq y} G_d(\tilde{y}) + \pi_{DF}), \inf_{\tilde{y} \geq y} G_d(\tilde{y}) + \frac{\pi_{DF}}{\pi_{\Delta}}\delta\right\}. \quad (9)$$

Any value outside of these bounds is clearly incompatible with the distribution of (Y, D, Z) and our assumptions. To illustrate the effect of our sensitivity parameters, we consider the width of these

¹¹We give an illustration of this derivation in Appendix C.8.

bounds for any fixed $y \in \mathbb{Y}$ as a function of (π_{DF}, δ) , that is¹² $\overline{H}_{Y_d^{CO}}(y, \pi_{DF}, \delta) - \underline{H}_{Y_d^{CO}}(y, \pi_{DF}, \delta)$. The width is weakly increasing in the sensitivity parameter δ , which implies that a larger violation of monotonicity leads to a larger identified set. However, the effect of the sensitivity parameter π_{DF} on this width can be both negative and positive depending on the specific underlying parameters θ . For example, we note that $F_{Y_d^{CO}}$ is point identified either if $\pi_{DF} = 0$ or $\pi_d = 0$, which denotes the absence of always or never takers. Heuristically speaking, the parameter π_{DF} , therefore, trades off the identification power gained from the non-existence of defiers and the non-existence of always or never takers.

The functions $\underline{H}_{Y_d^{CO}}$ and $\overline{H}_{Y_d^{CO}}$ clearly bound $F_{Y_d^{CO}}$ in a first-order stochastic dominance sense. However, since they do not imply that the implied functions of $F_{Y_d^{dT}}$ and $F_{Y_d^{DF}}$ are nondecreasing, they are not necessarily a feasible candidate of $F_{Y_d^{CO}}$. To give an intuition for this result and for the sake of argument, we now assume that all outcome variables are continuously distributed. We consider $\underline{H}_{Y_d^{CO}}$, and we assume that the bound on the outcome heterogeneity δ determines the bound, i.e., $\underline{H}_{Y_d^{CO}}(y) = G_d(y) - \frac{\pi_{DF}}{\pi_{\Delta}}\delta$. This bound does not necessarily imply that the always takers have a positive density. Specifically, the density of the lower bound is $g_d(y) = (q_{dd} - q_{d(1-d)}(y))/\pi_{\Delta}$, whereas to guarantee that the density function $f_{Y_d^{dT}}$ does not take any negative value, any feasible candidate of $f_{Y_d^{CO}}$ has to satisfy that

$$f_{Y_d^{CO}}(y) \leq \frac{q_{dd}(y)}{\pi_{CO}} \quad (10)$$

for all $y \in \mathbb{Y}$.¹³ A similar restriction as (10) can be derived for defiers such that any feasible candidate of the density $f_{Y_d^{CO}}(y)$ has to also satisfy that, for all $y \in \mathbb{Y}$,

$$f_{Y_d^{CO}}(y) \geq \frac{\pi_{\Delta}}{\pi_{CO}} \max\{g_d(y), 0\}. \quad (11)$$

Based on this argument, we construct our final bounds, $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$. Specifically, the distribution function $\underline{F}_{Y_d^{CO}}$ is dominated by $\underline{H}_{Y_d^{CO}}$ in a first-order stochastic dominance sense, and the distribution function $\overline{F}_{Y_d^{CO}}$ dominates $\overline{H}_{Y_d^{CO}}$ in a first-order stochastic dominance sense, and they both carefully take into account the reasoning of (11) and (10). In Appendix B.1, we show that these distribution functions both bound the distribution function $F_{Y_d^{CO}}$ and are feasible candidates.

¹²This comparison is helpful as the qualitative size of the width of the bounds on the distribution functions is related to the width of the identified set of many parameters of interest, e.g., the LATE.

¹³To be precise, one can assume that $q_{d(1-d)}(y) = 0$ and as $\pi_{\Delta} = \pi_{CO} - \pi_{DF} \leq \pi_{CO}$ the claim follows.

4.3. Identification Result. We first provide the analytical expressions of the bounds in the following. The lower bound of the distribution functions $F_{Y_d^{CO}}$ is given by

$$\begin{aligned} \underline{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta) &= \frac{1}{\pi_{CO}} Q_{dd}(y) \\ &\quad - \frac{1}{\pi_{CO}} \inf_{\tilde{y} \geq y} \left(Q_{dd}(\tilde{y}) - \left(\pi_{\Delta} G_d^+(\tilde{y}) - \inf_{\hat{y} \leq \tilde{y}} \left(\pi_{\Delta} G_d^+(\hat{y}) - \pi_{CO} \underline{H}_{Y_d^{CO}}(\hat{y}, \pi_{DF}, \delta) \right) \right) \right), \end{aligned} \quad (12)$$

and similarly the upper bound by

$$\begin{aligned} \overline{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta) &= \frac{\pi_{\Delta}}{\pi_{CO}} G_d^+(y) \\ &\quad - \frac{1}{\pi_{CO}} \sup_{\tilde{y} \geq y} \left(\pi_{\Delta} G_d^+(\tilde{y}) - \left(Q_{dd}(\tilde{y}) - \sup_{\hat{y} \leq \tilde{y}} \left(Q_{dd}(\hat{y}) - \pi_{CO} \overline{H}_{Y_d^{CO}}(\hat{y}, \pi_{DF}, \delta) \right) \right) \right). \end{aligned} \quad (13)$$

Based on the derivation above, Theorem 1 summarizes the result.

Theorem 1. *Suppose that Assumption 1 holds, and the data generating process is compatible with the sensitivity parameters (π_{DF}, δ) . Then, it holds that*

$$\underline{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta) \leq F_{Y_d^{CO}}(y) \leq \overline{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta),$$

for $d \in \{0, 1\}$ and for all $y \in \mathbb{Y}$. Moreover, there exist data generating processes which are compatible with the above assumptions such that the outcome distribution of compliers equals either $\underline{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta)$, $\overline{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta)$, or any convex combination of these bounds.

Theorem 1 shows not only that the proposed bounds are valid but also that without imposing further assumptions, the bounds cannot be tightened in a first-order stochastic dominance sense.¹⁴

Remark 1. Theorem 1 does clearly not imply that all distribution functions that are bounded by the distribution functions $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$ are feasible candidates of the distribution function of $F_{Y_d^{CO}}$. The reason for that is that these functions do not necessarily imply nondecreasing distribution functions of the other groups. Since we are not interested in the distributions functions themselves but in parameters defined through the bounds, this result is sufficient to derive sharp bounds on the sensitivity and robust region for empirical conclusions about these parameters.

Remark 2. The parameter of interest is often not only the average treatment effect but also, e.g., quantile and distribution treatment effects. As Theorem 1 bounds the entire outcome distribution

¹⁴As the derived bounds are rather complicated, we propose simpler bounds for each of our sensitivity parameters in Appendix A.1. These bounds are possibly conservative. We explain how to evaluate in an empirical setting whether they are close to the sharp bounds derived in this section.

functions of compliers, in a first-order stochastic dominance sense, these treatment effects are identified as well and are sharp for many relevant parameters. We present them in Appendix A.2.

Remark 3. In empirical applications, researchers also often have access to pre-intervention covariates. In Appendix A.3, we show how these covariates can be exploited to reduce the size of the identified set of the distribution function $F_{Y_d^{CO}}$. These covariates can then be used to tighten the sensitivity and to enlarge the robust regions.

5. SENSITIVITY ANALYSIS

We present our main sensitivity analysis in this section.

5.1. Sensitivity Region. We derive the sensitivity region, which is the set of sensitivity parameter pairs for which a feasible candidate of the distribution function $F_{Y_d^{CO}}$ exists. Sensitivity parameters that lie in the complement of this set refute the model, and we, therefore, do not consider them further.¹⁵

5.1.1. Population Size of Defiers. We show that the population size of defiers is partially identified. We denote an upper bound by

$$\bar{\pi}_{DF} = \min\{\mathbb{P}(D = 1|Z = 0), \mathbb{P}(D = 0|Z = 1)\}. \quad (14)$$

The first element of the minimum represents the sum of the population size of always takers and defiers, whereas the second one of never takers and defiers. The population size of defiers is clearly smaller than both of these quantities.

The lower bound on the population size of defiers is denoted by

$$\underline{\pi}_{DF} = \max_{s \in \{0,1\}} \left\{ \sup_{B \in \mathcal{B}} \{ \mathbb{P}(Y \in B, D = s|Z = 1 - s) - \mathbb{P}(Y \in B, D = s|Z = s) \} \right\}. \quad (15)$$

The supremum is taken over the differences in population distributions of defiers and compliers, which bounds the population size of defiers from below. The Proposition 1 shows that these bounds are sharp.¹⁶

Proposition 1. *Suppose Assumption 1 holds. Then the population size of defiers, π_{DF} , is bounded by $[\underline{\pi}_{DF}, \bar{\pi}_{DF}]$. Moreover, there exist data generating processes which are compatible with the above assumptions such that the population size of defiers equals any value within these bounds. Thus, the bounds are sharp.*

¹⁵Masten and Poirier (2021) denote the complement of the sensitivity region the falsification region.

¹⁶Richardson and Robins (2010) present sharp bounds on π_{DF} for a binary outcome variable.

If the lower bound on population size of defiers is greater than zero, $\underline{\pi}_{DF} > 0$, at least one of the classical LATE assumptions, including monotonicity, is violated.

This reasoning is align with the result of Kitagawa (2015), who shows that $\underline{\pi}_{DF} > 0$ is sufficient and necessary such that the LATE assumptions are valid.

However, if the above inequalities contradict, i.e., $\underline{\pi}_{DF} > \bar{\pi}_{DF}$, the sensitivity region is empty. This implies that even if one allows for a violation of monotonicity, our model assumptions must be violated as well.

5.1.2. *Outcome Heterogeneity.* We now consider the sensitivity parameter δ . Based on Theorem 1, we can bound the sensitivity parameter δ from below and from above for a given value of the sensitivity parameter π_{DF} .

A given pair fo sensitivity parameters (π_{DF}, δ) is refuted if the implied lower and upper bounds, $\underline{F}_{Y_d^{CO}}$ and $\bar{F}_{Y_d^{CO}}$, intersect, so that there does not exists a feasible candidate of the distribution function $F_{Y_d^{CO}}$ which is compatible with these sensitivity parameters. The domain of the sensitivity parameter δ is bounded from below by

$$\underline{\delta}(\pi_{DF}) = \min_{d \in \{0,1\}} \inf \{ \delta : \inf_y \underline{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta) - \bar{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta) \geq 0 \}. \quad (16)$$

The feasible set of the sensitivity parameter δ is further bounded from above. The bounds $\underline{F}_{Y_d^{CO}}$ and $\bar{F}_{Y_d^{CO}}$ imply bounds on the distribution function of $F_{Y_d^{DF}}$, where the largest value of the Kolmogorov-Smirnov norm between the distributions of $F_{Y_d^{CO}}$ and $F_{Y_d^{DF}}$ is achieved when $\delta = 1$. It follows that there does not exists a feasible candidate function of $F_{Y_d^{CO}}$ such that the implied outcome heterogeneity parameter exceeds this value. We denote the upper bounds by

$$\bar{\delta}(\pi_{DF}) = \max_{d \in \{0,1\}} \sup_{y \in \mathbb{Y}} \left\{ |\bar{F}_{Y_d^{CO}}(y, \pi_{DF}, 1) - \bar{F}_{Y_d^{DF}}(y, \pi_{DF}, 1)|, \right. \\ \left. |\underline{F}_{Y_d^{CO}}(y, \pi_{DF}, 1) - \underline{F}_{Y_d^{DF}}(y, \pi_{DF}, 1)| \right\}. \quad (17)$$

By the reasoning of Theorem 1, these bounds are sharp, and any convex combination of these bounds is feasible as well. It follows that our sensitivity region is given by

$$SR = \{ (\pi_{DF}, \delta) : \pi \in [\underline{\pi}_{DF}, \bar{\pi}_{DF}] \text{ and } \underline{\delta}(\pi_{DF}) \leq \delta \leq \bar{\delta}(\pi_{DF}) \}. \quad (18)$$

5.2. **Robust Region.** We now derive the robust region for the empirical conclusion that $\Delta_{CO} \geq \mu$.¹⁷ To simplify the presentation, we assume in the following that the sensitivity region is nonempty

¹⁷In Appendix A.2, we also consider other treatment effects than the average treatment effect of compliers. Sensitivity and robust regions for empirical conclusions about these parameters can then also be derived based on the reasoning of this section.

and that $\Delta_{CO}(\underline{\pi}_{DF}, \underline{\delta}(\underline{\pi}_{DF})) \geq \mu$.¹⁸

By first-order stochastic dominance of the distribution functions $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$, we can construct sharp bounds on many treatment effect parameters, that depend on these bounds (see Lemma 1 in Stoye, 2010). Specifically, let

$$\underline{\Delta}_{CO}(\pi_{DF}, \delta) = \int_{\mathbb{Y}} y d\overline{F}_{Y_1^{CO}}(y, \pi_{DF}, \delta) - \int_{\mathbb{Y}} y d\underline{F}_{Y_0^{CO}}(y, \pi_{DF}, \delta) \quad (19)$$

$$\overline{\Delta}_{CO}(\pi_{DF}, \delta) = \int_{\mathbb{Y}} y d\underline{F}_{Y_1^{CO}}(y, \pi_{DF}, \delta) - \int_{\mathbb{Y}} y d\overline{F}_{Y_0^{CO}}(y, \pi_{DF}, \delta). \quad (20)$$

Corollary 1. *Suppose that Assumption 1 holds, and the data generating process is compatible with the sensitivity parameters (π_{DF}, δ) . Then, the average treatment effect of compliers, Δ_{CO} , is bounded by $[\underline{\Delta}_{CO}(\pi_{DF}, \delta), \overline{\Delta}_{CO}(\pi_{DF}, \delta)]$. Moreover, there exist data generating processes which are compatible with the above assumptions such that the average treatment effect of compliers equals any value within these bounds. Thus, the bounds are sharp.*

For a given sensitivity parameter π_{CO} , we now consider the breakdown point given by

$$BP(\pi_{DF}) = \sup\{\delta : (\pi_{DF}, \delta) \in \text{SR and } \underline{\Delta}_{CO}(\pi_{DF}, \delta) \geq \mu\}.$$

For a given sensitivity parameter π_{DF} , it identifies the weakest assumption on outcome heterogeneity between compliers and defiers such that the empirical conclusion holds. The breakdown point, as a function of the sensitivity parameter π_{DF} , is not necessarily decreasing in the population size of defiers as the bounds on the outcome distribution of compliers can become tighter if the value of π_{DF} increases (see the discussion in Section 4.2). The breakdown frontier of the average treatment effect is the boundary of the robust region and given by the set of all breakdown points

$$BF = \{(\pi_{DF}, \delta) \in \text{SR} : \delta = BP(\pi_{DF})\}. \quad (21)$$

The robust region of the empirical conclusion that $\Delta_{CO} \geq \mu$ is characterized by

$$RR = \{(\pi_{DF}, \delta) \in \text{SR} : \delta \leq BP(\pi_{DF})\}. \quad (22)$$

The nonrobust region, that is the complement of the robust region within the sensitivity region, contains pairs of sensitivity parameters which only may not imply treatment effects being consistent with the empirical conclusion. Figure 2 illustrates one example of the sensitivity and robust region.¹⁹

¹⁸If $\Delta_{CO}(\underline{\pi}_{DF}, \underline{\delta}(\underline{\pi}_{DF})) < \mu$, the robust region is empty.

¹⁹We refer to a discussion on how these sets can be used in an empirical setting to Section 2 and 9

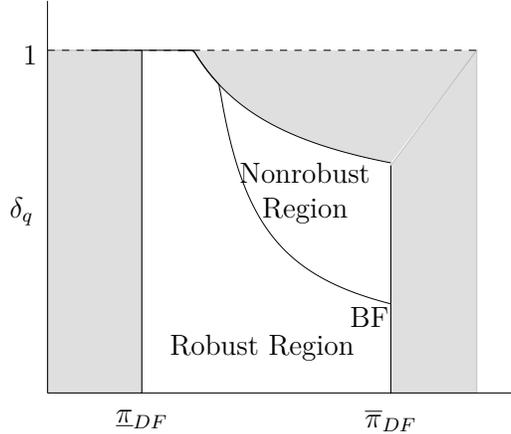


Figure 2: Sensitivity and Robust Region. Non-shaded region represents sensitivity region.

6. EXTENSIONS

In this section, we show how our framework can be exploited to draw empirical conclusions about other population parameters, and how it simplifies if the outcome variable is binary.

6.1. Treatment Effects for other Populations. To show how empirical questions about treatment effects of the entire population can be analyzed, we exploit that the proof of Theorem 1 presents sharp bounds on all groups in a first-order stochastic dominance sense. For $d \in \{0, 1\}$, let the lower bound be denoted by

$$\underline{E}_{Y_d}(y, \pi_{DF}, \delta) = \pi_{CO} \cdot \underline{E}_{Y_0^{CO}}(y, \pi_{DF}, \delta) + Q_{d(1-d)}(y),$$

and the upper bound by

$$\overline{F}_{Y_d}(y, \pi_{DF}, \delta) = \pi_d + \pi_{CO} \cdot \overline{F}_{Y_0^{CO}}(y, \pi_{DF}, \delta) + Q_{d(1-d)}(y).$$

Proposition 2. *Suppose the instrument satisfies Assumption 1, and the data generating process is compatible with the sensitivity parameters (π_{DF}, δ) . Then, it holds that*

$$\underline{E}_{Y_d}(y, \pi_{DF}, \delta) \leq F_{Y_d}(y, \pi_{DF}, \delta) \leq \overline{F}_{Y_d}(y, \pi_{DF}, \delta)$$

for $d \in \{0, 1\}$ and for all $y \in \mathbb{Y}$. Moreover, there exist data generating processes which are compatible with the above assumptions such that the potential outcome distributions equal either $\overline{F}_{Y_d}(y, \pi_{DF}, \delta)$, $\underline{E}_{Y_d}(y, \pi_{DF}, \delta)$, or any convex combination of these bounds.

As the data do not contain any information about the distribution functions $F_{Y_0^{AT}}$ and $F_{Y_1^{NT}}$, the bounds \underline{E}_{Y_d} and \overline{F}_{Y_d} are such that their respective probability mass is shifted to the extreme of

the support \mathbb{Y} . To interpret these bounds, for any y in the interior of \mathbb{Y} , we consider the difference

$$\overline{F}_{Y_d}(y, \pi_{DF}, \delta) - \underline{F}_{Y_d}(y, \pi_{DF}, \delta) = \pi_d.$$

The sensitivity parameter δ does not influence the distribution of the outcome of the entire population, as it only influences how the observed outcome probability mass is distributed between the groups. However, the size of the bounds decreases with the population size of defiers, π_{DF} , as the population size of always and never takers π_d decreases with π_{DF} . This reasoning is intuitive as if π_d decreases, the observed outcome probability mass represent more of the population under consideration. This result aligns with results of the literature Kamat (2018); Kitagawa (2021), who shows that imposing monotonicity (e.g., $\pi_{DF} = 0$) does not imply a smaller identified set of the average treatment effect of the entire population if the LATE assumptions are not violated.

Based on the bounds presented in Proposition 2, we can now derive a sensitivity analysis similar to the one presented in Section 5. However, to derive informative results about the average treatment effect of the entire population, we would have to impose that the outcome is bounded as otherwise the average treatment effect is not identified in general.

The sensitivity analysis of this paper is based on the premise that the treatment effect of compliers is the object of interest. However, if the parameter of interest is the treatment effect of the entire population, one might then be willing to impose assumptions not only on outcome heterogeneity between compliers and defiers but also between other groups. To be precise, we can replace the sensitivity parameter δ by $\delta_p \in [0, 1]$ such that

$$\max_d \sup_y \{|F_{Y_d^T}(y) - F_{Y_d^{T'}}(y)|\} \leq \delta_p \quad \forall T, T' \in \{AT, NT, CO, DF\}.$$

Using similar arguments as in the proof of Theorem 1, one can then derive sharp bounds on the outcome distribution functions of the entire population and then conduct a sensitivity analysis similar to the one described in Section 5. Empirical conclusions drawn on this parameterization might be substantially more informative.

6.2. Binary Outcome Variable. In many empirical applications, the outcome of interest is binary. The results of Section 4 and 5 are still valid in this case, but we show in this section that the bounds substantially simplify so that they are easier applicable. Let $P_d^T = \mathbb{P}(Y_d^T = 1)$ denote the probability that the random variable Y_d^T equals one, and let the conditional joint probability of the outcome and the treatment status be given by $P_{ds} = \mathbb{P}(Y = 1, D = d | Z = s)$. We denote the underlying parameters by $\theta_b = (P_{11}, P_{10}, P_{01}, P_{00}, P_0, P_1) \in [0, 1]^6$.

Following the same arguments as above, the sensitivity and robust region depend on the marginal

outcome distributions of the compliers. The presence of defiers is also bounded by π_{DF} , and the parameter of outcome heterogeneity simplifies to

$$\delta_b = \max_{d \in \{0,1\}} |P_d^{CO} - P_d^{DF}|.$$

The outcome probabilities of compliers are bounded from below by

$$\underline{P}_d^{CO}(\pi_{DF}, \delta) = \max \left\{ 0, \frac{P_{dd} - \pi_d}{\pi_{CO}}, \frac{P_{dd} - P_{d(1-d)}}{\pi_{CO}}, \frac{P_{dd} - P_{d(1-d)} - \pi_{DF}\delta_b}{\pi_{\Delta}} \right\}, \quad (23)$$

and from above by

$$\overline{P}_d^{CO}(\pi_{DF}, \delta) = \min \left\{ 1, \frac{P_{dd}}{\pi_{CO}}, \frac{P_{dd} - P_{d(1-d)} + \pi_{DF}}{\pi_{CO}}, \frac{P_{dd} - P_{d(1-d)} + \pi_{DF}\delta_b}{\pi_{\Delta}} \right\}. \quad (24)$$

Corollary 2. *Suppose Assumption 1 holds, and the data generating process is compatible with the sensitivity parameters (π_{DF}, δ) . The outcome probabilities of compliers are bounded by $\underline{P}_d^{CO} \leq P_d^{CO} \leq \overline{P}_d^{CO}$. Moreover, there exist data generating processes which are compatible with the above assumptions such that the population size of defiers equals any value within these bounds. Thus, the bounds are sharp.*

The interpretation of the width of these bounds follows the same reasoning as in Section 4.2. The lower bound of the population size of defiers simplifies to

$$\underline{\pi}_{DF} = \max_{d \in \{0,1\}} \left\{ \sum_{y=0}^1 \max\{0, \mathbb{P}(Y = y, D = d | Z = 1 - d) - P(Y = y, D = d | Z = d)\} \right\}.$$

The upper bound on π_{DF} cannot be simplified further and is given by (14). The lower bound on outcome heterogeneity is given by

$$\underline{\delta}_b(\pi_{DF}) = \frac{\underline{\pi}_{DF}}{\pi_{DF}}.$$

The lower bound on the sensitivity parameter δ decreases with the population size of defiers. The upper bound on the sensitivity parameter δ is given by the maximal difference between the outcome probabilities of compliers and defiers

$$\overline{\delta}_b(\pi_{DF}) = \max_{d \in \{0,1\}} \max\{|\underline{P}_d^{CO}(\pi_{DF}, 1) - \underline{P}_d^{DF}(\pi_{DF}, 1)|, |\overline{P}_d^{CO}(\pi_{DF}, 1) - \overline{P}_d^{DF}(\pi_{DF}, 1)|\}.$$

The sensitivity parameter space is given by

$$\text{SR}_b = \{(\pi_{\text{DF}}, \delta_b) \in [\underline{\pi}_{\text{DF}}, \bar{\pi}_{\text{DF}}] \times [0, 1] : \underline{\delta}_b(\pi_{\text{DF}}) \leq \delta_b \leq \bar{\delta}_b(\pi_{\text{DF}})\},$$

and the robust region for the claim $\Delta_{CO} \geq \mu$, if $\underline{P}_1^{CO}(\underline{\pi}_{\text{DF}}, \delta_b) - \bar{P}_0^{CO}(\underline{\pi}_{\text{DF}}, \delta_b) \geq \mu$, is given by

$$\text{RR}_b = \{(\pi_{\text{DF}}, \delta_b) \in \text{SR}_b : \underline{P}_1^{CO}(\pi_{\text{DF}}, \delta_b) - \bar{P}_0^{CO}(\pi_{\text{DF}}, \delta_b) \geq \mu\}.$$

Using the simple algebra structure of the bounds of the outcome probabilities, a closed-form expression for both the robust and the sensitivity region can be derived. As this expression is rather lengthy without providing much intuition, we state it in Appendix B.5.

7. ESTIMATION AND INFERENCE

Even though the contribution of this paper is the derivation of the sensitivity and robust region for a particular empirical conclusion, we consider some methods for estimation and inference of these two regions. While the technical details are deferred to Appendix C, in this section, we sketch the main issues of conducting inference in this setting and our proposed solutions. To simplify the exposition, we consider the setting of a continuous and a binary outcome variable, but our method is not restricted to these distributions.

Throughout this section, we assume that we have access to the data $\{(Y_i^z, D_i^z)\}_{i=1}^{n_z}$ for $z \in \{0, 1\}$ that are independent and identically distributed according to the distribution of (Y, D) conditionally on $Z = z$ with support $\mathbb{Y} \times \{0, 1\}$. We denote this distribution by (Y^z, D^z) and we let $n = n_0 + n_1$, where n_0/n converges to a nonzero constant as $n \rightarrow \infty$.²⁰

7.1. Estimation. To construct estimators of the sensitivity and robust region for a particular empirical conclusion, we note that the identification argument of these regions are constructive. It follows from Section 5 that the boundaries of both regions are identified by the following mapping,²¹

$$\phi(\theta, \pi_{\text{DF}}) = (\underline{\pi}_{\text{DF}}, -\bar{\pi}_{\text{DF}}, \underline{\delta}(\pi_{\text{DF}}), -\bar{\delta}(\pi_{\text{DF}}), BP(\pi_{\text{DF}})), \quad (25)$$

which is evaluated at the sensitivity parameter $\pi_{\text{DF}} \in [0, 0.5)$ and the underlying parameters θ , that is defined in (4). Estimating the sensitivity and robust region is then equivalent to estimating this mapping. To do so, we consider estimates of the underlying parameters θ that are simply obtained by replacing unknown population quantities by their corresponding nonparametric sample counterparts and by standard nonparametric kernel methods. We denote the estimates of θ by $\hat{\theta}$.

²⁰We discuss this assumption in Assumption C.1.

²¹The signs of the components of the mapping $\phi(\theta, \pi_{\text{DF}})$ simplify the subsequent analysis.

Point estimates of the mapping $\phi(\theta, \pi_{\text{DF}})$ can then be derived by simple plug-in methods. We defer a detailed description to Appendix C.4.

7.2. Goal of Inference. We propose to construct confidence sets for the sensitivity and robust region such that the confidence set for the sensitivity region is an outer confidence set and for the robust region is an inner confidence set.²² These confidence sets should therefore jointly satisfy with probability approaching the confidence level, $1 - \alpha$, that (i) any sensitivity parameter pair of the sensitivity region lies within the confidence set for the sensitivity region and (ii) not any single parameter pair of the nonrobust region lies within the confidence set for the robust region.²³ Let $\widehat{\text{SR}}_L$ and \widehat{RR}_L denote two sets of the sensitivity parameters. They satisfy the described condition if

$$\lim_{n \rightarrow \infty} \mathbb{P}(\text{SR} \subseteq \widehat{\text{SR}}_L \text{ and } \widehat{RR}_L(\text{SR}) \subseteq RR(\text{SR})) \geq 1 - \alpha. \quad (26)$$

Based on the definition of the mapping $\phi(\theta, \pi_{\text{DF}})$, it therefore suffices to construct a lower confidence band for each component of the estimator $\phi(\widehat{\theta}, \pi_{\text{DF}})$ as a function of π_{DF} that are jointly valid.²⁴ That is, we need to find a function that is componentwise a uniformly lower bound $\phi_L(\widehat{\theta}, \pi_{\text{DF}})$ in π_{DF} of $\phi(\theta, \pi_{\text{DF}})$ so that²⁵

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\min_{1 \leq l \leq 5} \inf_{\pi_{\text{DF}} \in [0, 0.5]} e_l^\top (\phi_L(\widehat{\theta}, \pi_{\text{DF}}) - \phi(\theta, \pi_{\text{DF}})) \leq 0 \right) \geq 1 - \alpha, \quad (27)$$

where e_l is the l -th unit vector.²⁶

7.3. Inference for a Continuous Outcome Variable. We analyze the distribution of $\phi(\widehat{\theta}, \pi_{\text{DF}})$ in order to construct confidence sets for the mapping $\phi(\theta, \pi_{\text{DF}})$.²⁷ Under regularity assumptions presented in Appendix C.2, the estimators of the underlying parameters $\widehat{\theta}$ converge in \sqrt{n} to a tight Gaussian process. Since the mapping ϕ is not Hadamard-differentiable, as it depends on minimum, maximum, supremum, and infimum of random functions, standard Delta method arguments do not

²²Considering inner confidence set for the robust region follows from Masten and Poirier (2020).

²³To give one more interpretation of the confidence sets and using the language of hypothesis testing, a sensitivity parameter pair, $(\pi_{\text{DF}}, \delta)$, does not lie in the sensitivity region only if we can reject such a hypothesis with confidence level $1 - \alpha$. Contrary, $(\pi_{\text{DF}}, \delta)$ lies in the robust region, only if we can reject that it is nonrobust with confidence level $1 - \alpha$. The confidence sets are constructed so that the hypothesis tests are valid uniformly in the sensitivity parameter space.

²⁴Throughout this section, we consider confidence sets that are uniformly valid in the sensitivity parameter space, but not necessarily in the distribution of the underlying parameters θ

²⁵We verify this equivalence in Appendix B.7.2.

²⁶Conservative confidence sets for only the average treatment effect of compliers for specific values of $(\pi_{\text{DF}}, \delta)$ directly follow from the presented procedure. To obtain nonconservative confidence sets, one can follow the literature on partially identified parameters (see, e.g., Imbens and Manski, 2004)

²⁷We want to emphasize that this procedure is valid for a fixed distribution. In particular, we do not consider settings of weak instruments or data generating processes which are such that the robust region becomes empty.

apply in this setup (see Fang and Santos, 2018). We propose a method to construct confidence sets that are asymptotically conservative but valid in the sense of (26). It is based on ideas of population smoothing that have been suggested by, e.g., Chernozhukov et al. (2010); Haile and Tamer (2003); Masten and Poirier (2020).

In contrast to considering the mapping ϕ , which identifies the sensitivity and robust region, we construct a smooth mapping, ϕ_κ , which yields valid bounds of both regions. The smoothed mapping ϕ_κ is indexed by a fixed smoothing parameter $\kappa \in \mathbb{N}$. The mapping ϕ_κ is differentiable such that the standard functional Delta method can be applied to ϕ_κ and we can study its asymptotic distribution by standard methods. The mapping ϕ_κ is further such that it yields an outer set of the sensitivity region and an inner set of the robust region. This reasoning implies that confidence sets of the smooth mappings ϕ_κ , which are valid in the sense of (26), are also valid for the mapping ϕ .

In finite samples, the choice of the smoothing parameter κ comprises the trade-off of constructing conservative confidence sets and better finite sample approximations of the underlying distributions. Suppose the smoothing parameter κ is small. In that case, the smoothed sensitivity and robust region are very similar to the original regions. However, the finite-sample distribution of $\phi_\kappa(\widehat{\theta}\pi_{\text{DF}})$ might not be well-approximated by its asymptotic distribution. Vice versa, suppose the smoothing parameter κ is large. The finite-sample distribution of $\phi_\kappa(\widehat{\theta}\pi_{\text{DF}})$ might be well-approximated by its asymptotic distribution. However, the smoothed sensitivity and robust region are conservative to the original regions.

In Appendix C.7.1, we show how the smoothed mappings can be constructed. It then follows that plug-in estimators of the smoothed mappings converge in \sqrt{n} to a Gaussian process by standard functional Delta method arguments. The covariance structure of this process is, in general, rather complicated and tedious to estimate. We, therefore, apply the nonparametric bootstrap to simulate its distribution. Consistency of this bootstrap procedure follows from arguments of Fang and Santos (2018). In Appendix C, we show how to construct the confidence sets based on the described procedure and that they achieve the outlined goal (26).

7.4. Inference for a Binary Outcome Variable. Following the discussion about a binary outcome model in Section 6.2, the mapping yielding the sensitivity and the robust region for a particular conclusion for a binary outcome variable is given by²⁸

$$\phi_b(\theta_b, \pi_{\text{DF}}) = (\underline{\pi}_{\text{DF},b}, -\bar{\pi}_{\text{DF},b}, -\bar{\delta}_b(\pi_{\text{DF}}), BP_b(\pi_{\text{DF}})),$$

²⁸where its precise definition follows from Section 6.2 and Appendix C.3.

The interpretation of ϕ_b follows the one for a continuously distributed outcome variable, and in principle, we could apply the same inference procedure as described above. However, the mapping $\phi_b(\theta_b, \pi_{\text{DF}})$ is substantially simpler than the mapping $\phi(\theta, \pi_{\text{DF}})$ so that, in this section, we can apply more classical inference procedure to obtain confidence sets in the sense of (26); in particular, we follow ideas of Masten and Poirier (2020) and the literature about moment inequalities (see, e.g., Andrews and Soares, 2010).

Under standard sampling assumptions, it follows that the estimators of the underlying parameters are jointly \sqrt{n} normally distributed (see Appendix C.3). The mapping $\phi_b(\theta_b, \pi_{\text{DF}})$ is clearly not Hadamard-differentiable, as it consist of minimum and maximum of random functions. Standard Delta method arguments are therefore not applicable here as well. Valid confidence sets could be obtained by projection arguments, which, however, are known to be conservative in general.

We show instead that the mapping ϕ_b is Hadamard directionally differentiable in the direction of θ when evaluated at finitely many $\{\pi_{\text{DF}}^k\}_{k=1}^K$. Using generalized Delta method arguments, the estimator of the mapping ϕ_b converges to some tight random process, which is a continuous transformation of a Gaussian process, indexed at the finite set $\{\pi_{\text{DF}}^k\}_{k=1}^K$. As this limiting distribution is rather complicated, we do not construct our inference procedure directly on its limiting distribution, but one can choose various modified bootstrap methods to simulate this distribution, e.g., subsampling or numerical-Delta method (see Dümbgen, 1993; Hong and Li, 2018). In this paper, we follow a bootstrap method which relies on ideas based on the moment inequality literature (see, e.g., Andrews and Soares, 2010; Bugni, 2010) and we explain the procedure in detail in Appendix C.3. Based on this bootstrap procedure, we can construct valid lower confidence sets for ϕ_b indexed at the finite set of sensitivity parameters $\{\pi_{\text{DF}}^k\}_{k=1}^K$. Using these confidence sets and exploiting the functional form of ϕ_b , we then obtain lower confidence sets for the estimator of the mapping ϕ_b , which are uniformly valid in π_{DF} . We state these arguments precisely in Appendix C.3 and show that these confidence sets are asymptotically valid in the sense of our goal of inference (26).

8. SIMULATIONS

8.1. Setup. We study the finite sample performance of the proposed estimators of the sensitivity and robust regions through a Monte Carlo study. We consider different data generating processes with varying degrees of violations of monotonicity, implying different sizes and shapes of both the sensitivity and robust regions. Specifically, we consider the following population sizes $(\pi_{\text{CO}}, \pi_{\text{DF}}) \in \{(0.35, 0.05), (0.25, 0.15)\}$, where $\pi_{\text{DF}} = \pi_{\text{AT}} = 0.3$. We set $\mathbb{P}(Z = 1) = 0.5$ and we generate the

Table 1: Simulated coverage rates of the sensitivity and robust region for a positive treatment effect.

π_{CO}	Δ_{CO}	Δ_{TE}	$\eta = 0.2$	$\eta = 0.5$	$\eta = 1$	$\eta = 1.5$	$\eta = 2$
0.35	0.3	0	99.1	97.8	95.1	91.3	90.8
	0.3	-0.3	96.9	94.3	91.2	92.5	91.6
	0.1	0	99.3	98.5	96.0	92.9	91.2
	0.1	-0.3	99.3	98.5	95.6	93.1	91.3
0.25	0.3	0	98.9	98.1	95.2	92.4	91.2
	0.3	-0.3	99.1	98.0	94.3	92.9	91.4
	0.1	0	99.3	98.6	96.0	93.5	91.2
	0.1	-0.3	99.0	97.7	94.3	92.9	90.9

The data generating process and the expressions follow the description of the text. Results are based on 10,000 Monte Carlo draws.

outcome by

$$Y_1^{CO} \sim \mathcal{B}(1, 0.5 + \Delta_{CO}) \quad Y_1^{DF} \sim \mathcal{B}(1, 0.5 + \Delta_{DF}) \quad Y_1^{AT}, Y_0^{NT}, Y_0^{DF}, Y_0^{CO} \sim \mathcal{B}(1, 0.5),$$

where $\Delta_{CO} \in \{0.2, 0.1\}$, and $\mathcal{B}(1, p)$ denotes the Bernoulli distribution with parameter p . The sensitivity region is nonempty as the data generating process satisfies our model assumptions. We consider the empirical conclusion of a positive treatment effect of compliers, so that the robust region is nonempty in each of the data generating processes as the Wald estimand is positive. The bootstrap procedure requires to choose the tuning parameter η , which is explained in Appendix C.3. We consider different values of η given by $\{0.2, 0.5, 1, 1.5, 2\}/\sqrt{N}$. The results are based on 10,000 Monte Carlo draws.

8.2. Simulation Results. Table 1 shows the results of the simulated coverage rates at which the confidence sets cover the population sensitivity and nonrobust region for the different data generating processes and choices of tuning parameters. Our considered choice of tuning parameters implies that the simulated coverage of our confidence sets is close to the nominal one in most data generating processes. These results illustrate that the confidence method performs reasonably well in finite samples.

9. EMPIRICAL APPLICATION

To illustrate our proposed framework, we apply this sensitivity analysis to data from Angrist and Evans (1998), who analyze the effect of having a third child on the labor market outcomes of

mothers. It is shown that even small violations of the monotonicity assumption may have a large impact on the robustness of the estimated treatment effects such that even the sign of the treatment effects may be indeterminate. The same-sex instrument in Angrist and Evans (1998) arguably satisfies Assumption 1: The independence assumption seems to be plausible by the following reasoning: The sex of a child is determined by nature, and only the number of and not the sex of the child arguably influences the labor market outcome. The relevance assumption is testable. However, monotonicity might be violated. We apply the proposed sensitivity analysis to evaluate the robustness of the estimated treatment effects to a potential violation of monotonicity in this setting. For simplicity, we focus on two outcome variables: the labor market participation of mothers and their annual wage.²⁹ The binary decision to treat represents the extensive margin and the continuous outcome variable a mix of extensive and intensive variables. We use the same data as Angrist and Evans (1998).³⁰ The sample size is 211,983. The point estimated difference of the population sizes of compliers and defiers is given by 0.06.

9.1. Sensitivity Analysis for Binary Outcome Variable . We consider the labor market participation of mothers as the outcome variable. The Wald estimate is given by -0.13 . Figure 3 illustrates the 95% confidence set for the sensitivity and the robust region for the claim that the treatment effect of compliers is negative. The formal definition of these confidence sets is given in Section 7.2. In this example, a (conservative) 95% confidence set for π_{DF} is given by $[0, 0.37]$. Following the literature, one can therefore not conclude that monotonicity is violated in this example (see for a comparison, e.g., Small et al., 2017). The sensitivity parameter pairs below the red line represent the robust region, which is the estimated set of sensitivity parameters implying a negative treatment effect. This figure shows that concerns about the validity of the monotonicity assumption have to be taken seriously. Since $BP(0.37)$ is almost zero, the hypothesis that the treatment effect is negative cannot be rejected without imposing any assumptions on the data generating process, If the population size of defiers increases, the breakdown frontier is relatively steeply declining, and thus the robust region is rather small. This implies that relatively strong assumptions on the outcome distributions of compliers and defiers have to be imposed to conclude that the treatment effect is negative in the presence of defiers. In contrast, if the population size of defiers is small, it is not necessary to impose strong assumptions about heterogeneity in the outcome variables to imply a negative effect.

This example shows that without imposing any assumptions on the data generating process,

²⁹The annual wage is a continuously distributed running variable with a point mass at zero.

³⁰Data are taken from the website Joshua D. Angrist website www.economics.mit.edu/faculty/angrist from 1980. The sample is restricted to women at the age of 20-36, having at least two children, being white, and having their first child at the age of 19-25.

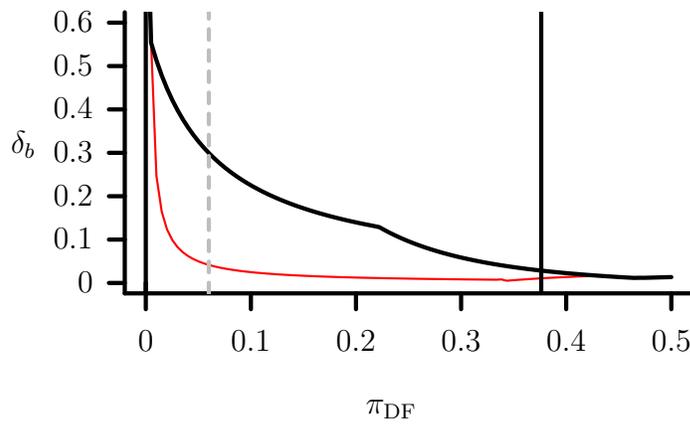


Figure 3: Confidence sets for the sensitivity and robust region for a negative treatment effect of compliers. The confidence level is 95 %. The treatment effect of compliers is the effect of having a third child on the labor market participation of mothers complying with the same-sex instrument. The black lines bound the sensitivity region, and the red line indicates the boundary of the robust region. The population size of defiers is on the horizontal axis, and outcome heterogeneity between compliers and defiers on the vertical axis.

only non-informative conclusions can be drawn in this example, which is the case as the population size of defiers is not much restricted and is arguably implausible high.³¹ One, therefore, might be willing to impose further assumptions to arrive at more interesting results, and we show how one could plausibly proceed. These assumptions should only serve as an example, and obviously, they have to be always adapted to the analyzed situation. We adopt the approach of De Chaisemartin (2017). One of the most essential inherently unknown quantities of interest is the population size of defiers. Imposing a smaller upper bound of this quantity based on economic reasoning allows us to derive sharper results. Based on a survey conducted in the US, De Chaisemartin (2017) states that it seems reasonable that 5% of defiers is a conservative upper bound of the population size of defiers in this setting. If one is willing to impose this assumption, one would still have to assume that the differences in the Kolmogorov-Smirnov norm are less than 0.05, which is a quite strong assumption. Therefore, we would conclude that the treatment effect is not robust to a potential violation in this specific example.

9.2. Sensitivity Analysis for Continuous Outcome Variable. We now consider the annual log income of the mother. This variable has a point mass at zero, representing all women who do not work but is otherwise continuously distributed. The Wald estimate is given by -1.23 . Figure 4 shows the corresponding 95% confidence sets for the robust and sensitivity region. If the monotonicity assumption were not violated, this estimate would imply that women who get a third child have an annual log wage reduced by 1.23.

Figure 4 shows 95 % confidence sets for both the sensitivity and robust region. The same line of interpretation applies as in the case of a binary outcome variable. One can see that without imposing any assumption about the population size of defiers, the empirical conclusion of a negative treatment effect is not robust to a potential violation of monotonicity. However, applying the same reasoning as above and imposing a maximal population size of 5% as an upper bound of the population size of defiers, one can see that the empirical conclusion is now robust to a potential violation of monotonicity.

To conclude, this sensitivity analysis is of interest, as one can identify the sign and the order of magnitude of the treatment effects by imposing further assumptions. These imposed assumptions are substantially weaker than the monotonicity assumption so that the estimates gain credibility.

³¹To interpret these numbers, we note that the upper bound is a rather conservative estimate. If roughly 37% of the population were a defier, then approximately 43% of the population would have been a complier. This reasoning implies that roughly 90% of the population would base their decision to have a third child on the sex composition of the first two children.

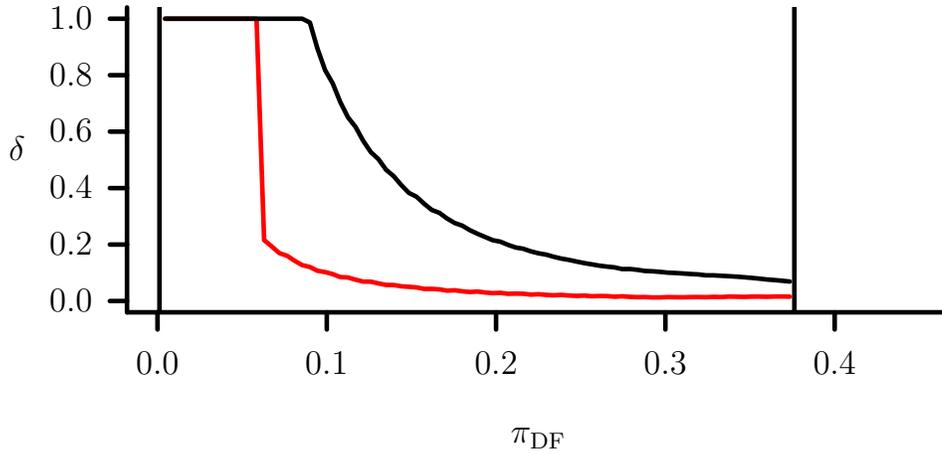


Figure 4: Confidence Sets for the Sensitivity and Robust Region for a Negative Treatment Effect of Compliers. The confidence level is 95 %. The compliers treatment effect is the effect of getting a third child on the annual log wage of mothers complying with the same-sex instrument. The black lines bound the sensitivity region, and the red line indicates the boundary of the robust region.

10. CONCLUSION

The local average treatment effect framework is popular to evaluate heterogeneous treatment effects in settings of endogenous treatment decisions and instrumental variables. In some empirical settings, one might doubt the validity of one of its key identifying assumptions, the monotonicity assumption. Conducting a sensitivity analysis of the estimates in these settings improves the reliability of the results. This paper, therefore, proposes a new framework, which allows researchers to assess the robustness of the treatment effect estimates to a potential violation of monotonicity. It parameterizes a violation of monotonicity by two parameters, the presence of defiers and heterogeneity of defiers and compliers. The former parameter is represented by the population size of defiers and the latter by the Kolmogorov-Smirnov norm bounding the outcome distributions of both groups. Based on these two parameters, we derive sharp identified sets for the average treatment effect of compliers and for any other group under further mild support assumption on the outcome variable. These identification results allow us to identify the sensitivity parameters that imply conclusions of treatment effect being consistent with the empirical conclusion. The empirical example of Angrist and Evans (1998) same-sex instruments underlines the importance of the validity of the monotonicity assumptions as small violations of monotonicity may already lead to uninformative results.

APPENDIX

A. ADDITIONAL MATERIALS FOR THE SENSITIVITY ANALYSIS

In this section, we collect additional materials on identification of the sensitivity region. In Section A.1, we present simplified bounds, and we consider additional treatment effects in Section A.2. We explain how covariates can be used to tighten the bounds in Section A.3 and we give further results on a binary outcome variable in Section A.4.

A.1. Simple Bounds. The bounds presented in Theorem 1 are rather tedious, and conducting inference for the sensitivity and robust regions based on these bounds is complicated as it depends on many tuning parameters to choose. In this section, we, therefore, present simpler bounds, which might be more easily applicable in an empirical context, and confidence sets of these regions might be more reliable. The proposed simple bounds are especially suited for settings in which the empirical researcher does not have evidence for the existence of defiers and expects that the number of defiers is, if any, smaller than of compliers. In this case, the simplified bounds on the distribution function $F_{Y_d^{CO}}$ are similar to the sharp bounds of Theorem 1.

The main reason for the rather complicated expression of the sharp bounds $\overline{F}_{Y_d^{CO}}$ and $\underline{F}_{Y_d^{CO}}$ is heuristically that these bounds exploit all the information contained about the defiers included in the function $G_d(y)$. However, if we forgo the aim of constructing sharp bounds, we can simply look at the functions

$$\underline{F}_{Y_d^{CO}}^S(y, \pi_{DF}, \delta) = \max \left\{ 0, \frac{1}{\pi_{CO}}(Q_{dd}(y) - \pi_d), \frac{\pi_{\Delta}}{\pi_{CO}}G_d(y), G_d(y) - \frac{\pi_{DF}}{\pi_{\Delta}}\delta \right\},$$

and

$$\overline{F}_{Y_d^{CO}}^S(y, \pi_{DF}, \delta) = \min \left\{ 1, \frac{1}{\pi_{CO}}Q_{dd}(y), \frac{\pi_{\Delta}}{\pi_{CO}}(G_d(y) + \pi_{DF}), G_d(y) + \frac{\pi_{DF}}{\pi_{\Delta}}\delta \right\}.$$

It follows from the reasoning of the main text that these functions satisfy that any value outside of these bounds is incompatible with the distribution of (Y, D, Z) , and our model assumptions.³² They are therefore always valid bounds of the distribution function $F_{Y_d^{CO}}$.

Our simplified sensitivity analysis is then based on the simplified bounds, and we further ignore the lower bound on the sensitivity parameters $\underline{\pi}_{DF}$ and $\underline{\delta}$ and set both to zero. These bounds are still valid for both the sensitivity and robust region. Estimation and conducting inference within this sensitivity analysis is substantially more straightforward.

In an empirical setting, it remains the question under which conditions these are actually "good"

³²This reasoning directly follows from the discussion of Section 4.

bounds in the sense that they are close to the sharp bounds and do not lead to a substantial loss in information. The most important difference between the sharp and these simple bounds is that we do not exploit the information about the defiers, which is contained in the function G_d for $d \in \{0, 1\}$ and the information that the distribution function is not allowed to increase too much. So if G_d is indeed decreasing, we expect that the sharp bounds are substantially more informative than the simple bounds. To assess how conservative these bounds might be, a researcher could also test whether $G_d(y)$ is non-decreasing for all $y \in \mathbb{Y}$ (Kitagawa, 2015).

A.2. Additional Treatment Effects. The sharp bounds on the distribution function, $F_{Y_d^{CO}}$, in a first-order stochastic dominance sense, allows us to consider various other treatment effects as well. In this section, we consider quantile treatment effects and define the τ -th quantile effect of the compliers by $\Delta_{CO}(\tau)$ and we consider empirical conclusions of the form $\Delta_{CO}(\tau) \geq \mu$.

We define the lower and upper bounds of the quantile functions by the respectively left and right inverse of the bounds of the outcome distributions

$$\begin{aligned} \underline{Q}_{Y_d^{CO}}(\tau, \pi_{DF}, \delta) &= \inf\{y \in \mathbb{Y} : \overline{F}_{Y_d^{DF}}(y, \pi_{DF}, \delta) \geq \tau\} \\ \overline{Q}_{Y_d^{CO}}(\tau, \pi_{DF}, \delta) &= \sup\{y \in \mathbb{Y} : \underline{F}_{Y_d^{DF}}(y, \pi_{DF}, \delta) \leq \tau\}. \end{aligned}$$

The quantile treatment effect of a quantile τ is then given by

$$\begin{aligned} &[\underline{\Delta}_{CO}(\tau, \pi_{DF}, \delta), \overline{\Delta}_{CO}(\tau, \pi_{DF}, \delta)] \\ &= [\underline{Q}_{Y_1^{CO}}(\tau, \pi_{DF}, \delta) - \overline{Q}_{Y_0^{CO}}(\tau, \pi_{DF}, \delta), \overline{Q}_{Y_1^{CO}}(\tau, \pi_{DF}, \delta) - \underline{Q}_{Y_0^{CO}}(\tau, \pi_{DF}, \delta)]. \end{aligned}$$

It follows from the reasoning of Lemma 1 in Stoye (2010) that these bounds are indeed sharp as well, and there exist feasible candidate distribution functions of $F_{Y_d^{CO}}$, which also imply any value between these bounds.

The sensitivity region is defined independently of the particular empirical conclusion under consideration and is therefore given by the expression of the main text (18). It follows that the breakdown point for the conclusion that $\Delta_{CO}(\tau) \geq \mu$ is given by

$$BP_\tau(\pi_{DF}) = \sup\{\delta : (\pi_{DF}, \delta) \in \text{SR} \text{ and } \underline{\Delta}_{CO}(\tau, \pi_{DF}, \delta) \geq \mu\}.$$

The breakdown frontier of the quantile treatment effect is given by

$$BF_\tau = \{(\pi_{DF}, \delta) \in : \delta = BP_\tau(\pi_{DF})\}.$$

and the robust region by

$$RR_\tau = \{(\pi_{DF}, \delta) \in SR : \delta \leq BP_\tau(\pi_{DF})\}.$$

A.3. Additional Covariates. Additional covariates, which are measured prior to treatment assignment, can be used to tighten the bounds on the identified set of treatment effects of compliers and can thus lead to greater sets of the robust region; the arguments are similar to those in Lee (2009). It further holds that conditioning on pretreatment covariates can imply that the identified set of the sensitivity parameters can be reduced such that the analysis becomes more informative. We, therefore, assume that the covariates are discrete and given by $\mathcal{X} = \{x_1, \dots, x_K\}$, which splits the population into non-overlapping groups. We further impose the following assumptions.

Assumption A.1. (i) *Conditional independence assignment:* $(Y_1, Y_0, D) \perp Z | X = x$, (ii) *Conditional relevance:* $\mathbb{P}(D = 1 | Z = 1, X = x) > \mathbb{P}(D = 1 | Z = 0, X = x)$, (iii) *Common support:* $0 < \mathbb{P}(Z = 1 | X = x) < 1$.

We denote the population size of defiers by $\pi_{DF}(x)$ given $X = x$. We consider the sensitivity parameter π_{DF} such that for all $x \in \{x_1, \dots, x_K\}$

$$\pi_{DF}(x) \leq \pi_{DF}.$$

This parameterization implies that the population size of defiers is bounded from above for each value of the covariates. We note that this parameterization implies without further assumptions conservative bounds as long as $\pi_{DF}(x) \neq \pi_{DF}$ for some values of x .³³

By similar reasoning the heterogeneity in the outcome distribution is restricted by

$$|F_{Y_d^{CO}|X=x}(y|X=x) - F_{Y_d^{DF}|X=x}(y|X=x)| \leq \delta_x.$$

Based on the pre-intervention covariates one can calculate for each k lower and upper bounds on the population size of defiers $\underline{\pi}_{DF}(x_k)$ and $\bar{\pi}_{DF}(x_k)$, respectively. The bounds on the sensitivity parameters can then be calculated based on the definition of the sensitivity parameters by $\underline{\pi}_{DF} = \min_{x \in \mathcal{X}}(\underline{\pi}_{DF}(x))$ and $\bar{\pi}_{DF} = \max_{x \in \mathcal{X}}(\bar{\pi}_{DF}(x))$. Let $\pi_{DF}(x_k) = \min\{\bar{\pi}_{DF}(x_k) \max\{\pi_{DF}, \underline{\pi}_{DF}(x_k)\}\}$

³³We consider two alternative parameterization: First, one could argue that $\pi_{DF}(x) = \pi_{DF}$ for all $x \in \mathcal{X}$. The implied bounds would be sharp, but this assumption is very restrictive. Second, we could consider a setting of $\pi_{DF}(x) = \pi_{DF}^x$. In this parameterization, however, the parameter space might be very large and therefore difficult to interpret. The parameterization chosen in the text is plausible and interpretable.

and $\pi_{\text{DF}}^x = \sum_{k=1}^K \pi_{\text{DF}}(x_k)$. We denote the lower bound by

$$\underline{F}_{Y_d^{\text{CO}}}^x(y, \pi_{\text{DF}}, \delta_x) = \frac{1}{\pi_{\text{DF}}^x} \sum_{k=1}^K \mathbb{P}(X = x_k, \pi_{\text{DF}}(x_k)) \underline{F}_{Y_d^{\text{CO}}}^x(y, \pi_{\text{DF}}(x_k), \delta_x | X = x_k)$$

and the upper bound by

$$\overline{F}_{Y_d^{\text{CO}}}^x(y, \pi_{\text{DF}}, \delta_x) = \frac{1}{\pi_{\text{DF}}^x} \sum_{k=1}^K \mathbb{P}(X = x_k, \pi_{\text{DF}}(x_k)) \overline{F}_{Y_d^{\text{CO}}}^x(y, \pi_{\text{DF}}(x_k), \delta_x | X = x_k).$$

Proposition A.1. *Suppose that Assumption A.1 holds, and the data generating process is compatible with the sensitivity parameters $(\pi_{\text{DF}}, \delta_x)$. Then, for $d \in \{0, 1\}$*

$$\underline{F}_{Y_d^{\text{CO}}}(y, \pi_{\text{DF}}, \delta_x) \leq F_{Y_d^{\text{CO}}}(y, \pi_{\text{DF}}, \delta_x) \leq \overline{F}_{Y_d}(y, \pi_{\text{DF}}, \delta_x).$$

Moreover, there exist data generating processes which are compatible with the model assumptions such that the outcome distribution of compliers equals either $\overline{F}_{Y_d^{\text{CO}}}^x(y, \pi_{\text{DF}}, \delta_x)$, $\underline{F}_{Y_d^{\text{CO}}}^x(y, \pi_{\text{DF}}, \delta_x)$, or any convex combination of these bounds, if $\pi_{\text{DF}}^x = \pi_{\text{DF}}$, and if for all $x \in \mathcal{X}$ and for $d \in \{0, 1\}$, it holds that

$$\sup_{y \in \mathbb{Y}} |F_{Y_d^{\text{CO}}|X=x}(y|X=x) - F_{Y_d^{\text{DF}}|X=x}(y|X=x)| = \delta.$$

The derivation of the sensitivity and robust region follows from the same arguments as in Section 5.

A.4. Form of Sensitivity and Robust Region for Binary Outcome Variable. Since our inference procedure exploits the shape of the sensitivity and robust region for a particular empirical conclusion about a binary outcome variable, we discuss this form in this section. We note again that they are determined by the following parameters.

$$\phi_b(\theta_b, \pi_{\text{DF}}) = (\underline{\pi}_{\text{DF}}, -\overline{\pi}_{\text{DF}}, -\overline{\delta}(\pi_{\text{DF}}), BP(\pi_{\text{DF}})),$$

We discuss the components in turn. The sensitivity region is determined based on four parameters: the lower and upper bound on the population size of defiers and the lower and upper bound on the sensitivity parameter of outcome heterogeneity. Due to their simple form, we do not have to discuss the lower and upper bound on the population size of defiers further, neither the lower bound on the outcome heterogeneity. However, the upper bound on the sensitivity parameter of outcome heterogeneity is given by

$$\overline{\delta}_b(\pi_{\text{DF}}) = \max_{d \in \{0, 1\}} \max\{|\underline{P}_d^{\text{CO}}(\pi_{\text{DF}}, 1) - \underline{P}_d^{\text{DF}}(\pi_{\text{DF}}, 1)|, |\overline{P}_d^{\text{CO}}(\pi_{\text{DF}}, 1) - \overline{P}_d^{\text{DF}}(\pi_{\text{DF}}, 1)|\}.$$

Following the discussion about how the bounds are constructed, e.g., in Appendix C.8, it follows that the upper bound on outcome heterogeneity has to be decreasing in the population size of defiers as the distribution of both compliers and defiers become more similar. We now consider the breakdown point as a function of the population size of defiers. We note that it can be rewritten as

$$BP(\pi_{\text{DF}}) = \frac{1}{\pi_{\text{DF}}} \max\{BP_0(\pi_{\text{DF}}), BP_1(\pi_{\text{DF}}), BP_2(\pi_{\text{DF}})\},$$

where $BP_0(\pi_{\text{DF}})$ is decreasing, $BP_1(\pi_{\text{DF}})$ and $BP_2(\pi_{\text{DF}})$ are potentially increasing so that

$$\begin{aligned} & BP_0(\pi_{\text{DF}}) \\ &= \max \left\{ P_{11} - P_{10} - \left(\mu + \frac{P_{00} - P_{01} + \pi_{\text{DF}}}{\pi_{\text{CO}}} \right) \pi_{\Delta}, - \left(\left(\mu - \frac{P_{11} - P_{10}}{\pi_{\text{CO}}} \right) \pi_{\Delta} + P_{00} - P_{01} \right) \right. \\ & \quad \left. - (\mu \cdot \pi_{\Delta} + P_{00} - P_{01}), P_{11} - P_{10} - (\mu + 1) \pi_{\Delta}, \frac{1}{2} (P_{11} - P_{10} - P_{00} + P_{01} - \mu \cdot \pi_{\Delta}), 0 \right\} \\ & BP_1(\pi_{\text{DF}}) = \max \left\{ 0, - \left(\left(\mu - \frac{P_{11} - \pi_{\text{AT}}}{\pi_{\text{CO}}} \right) \pi_{\Delta} + P_{00} - P_{01} \right) \right\} \\ & BP_2(\pi_{\text{DF}}) = \max \left\{ 0, P_{11} - P_{10} - \left(\mu + \frac{P_{00}}{\pi_{\text{CO}}} \right) \pi_{\Delta} \right\}. \end{aligned}$$

We therefore denote by

$$\tilde{\phi}_b(\theta_b, \pi_{\text{DF}}) = (\underline{\pi}_{\text{DF}}, -\bar{\pi}_{\text{DF}}, -\bar{\delta}(\pi_{\text{DF}}), BP_0(\pi_{\text{DF}}), BP_1(\pi_{\text{DF}}), BP_2(\pi_{\text{DF}})). \quad (28)$$

Each component of the mapping $\tilde{\phi}_b$ is either nondecreasing or nonincreasing in π_{DF} . We exploit this shape constraint to construct confidence sets for the sensitivity and robust region that are uniformly valid in π_{DF} .

B. PROOFS OF MAIN RESULTS

In this section, we prove the main results of this paper.

B.1. Proof of Theorem 1. As we consider a fix sensitivity parameter pair $(\pi_{\text{DF}}, \delta)$, we omit the dependence of all functions on the sensitivity parameter in this section; for instance, we write $\underline{F}_{Y_d^{\text{CO}}}(y)$ instead of $\underline{F}_{Y_d^{\text{CO}}}(y, \pi_{\text{DF}}, \delta)$.

We first consider how to determine whether a distribution function $F_{\tilde{Y}_d^{CO}}$ is indeed a feasible candidate of $F_{Y_d^{CO}}$. We therefore have to construct a random variable $\tilde{W} \equiv (\tilde{Y}_0, \tilde{Y}_1, \tilde{D}_0, \tilde{D}_1, \tilde{Z})$, that is compatible with the model assumptions, the observed probabilities and the sensitivity parameters, and implies $F_{\tilde{Y}_d^{CO}}$ as outcome distribution of compliers. However, based on our model assumptions, e.g. independence and the definition of the groups, it indeed suffices to construct marginal outcome distribution functions of

$$F_{\tilde{Y}_d^{CO}}, F_{\tilde{Y}_d^{DF}}, F_{\tilde{Y}_d^{AT}}, F_{\tilde{Y}_d^{NT}},$$

which are consistent with the observed probabilities and the sensitivity parameters. Since the data are also noninformative about the distribution functions $F_{Y_0^{AT}}$ and $F_{Y_1^{NT}}$ these distributions are left unrestricted as well. As outlined in the main text, any given candidate distribution function of $F_{Y_d^{CO}}$ implies functions of $F_{Y_d^{DF}}$ and $F_{Y_d^{DT}}$ given our sensitivity parameters and the observed distributions (Y, D, Z) . It follows that $F_{\tilde{Y}_d^{CO}}$ is a feasible candidate of $F_{Y_d^{CO}}$ if the implied functions of $F_{Y_d^{DF}}$ and $F_{Y_d^{AT}}$ are indeed distribution functions.

We argue in the main text that any feasible candidate of the distribution function $F_{Y_d^{CO}}$ has to satisfy at least that

$$\underline{H}_{Y_d^{CO}}(y) \leq F_{Y_d^{CO}}(y) \leq \overline{H}_{Y_d^{CO}}(y) \quad (29)$$

The proof now proceeds in two parts. In part I, we exploit which information can be obtained from the observed probabilities about the compliers outcome distribution to show which additional restriction, besides (29), any feasible candidate of $F_{Y_d^{CO}}$ has to satisfy. In part II, we then verify that the proposed bounds $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$ are feasible candidates of the distribution function $F_{Y_d^{CO}}$. We show that these bounds satisfy that any value outside of these bounds is incompatible with the distribution of (Y, D, Z) , and our assumptions. We denote by \underline{y} and \overline{y} the respectively left and right limits of \mathbb{Y} , which might equal $\pm\infty$.

Let $G_d^{\text{sup}}(y) = \sup_{\hat{y} \leq y} G_d(\hat{y})$ that is (i) a nondecreasing function and (ii) satisfying $G_d(y) \leq G_d^{\text{sup}}(y)$ for all $y \in \mathbb{Y}$. It further holds that $G_d^{\text{sup}}(y)$ is such that $G_d^{\text{sup}}(y) \leq \tilde{G}_d(y)$ for all $y \in \mathbb{Y}$, where \tilde{G}_d is any real-valued function \tilde{G}_d satisfying conditions (i) and (ii). Similarly, let $G_d^{\text{inf}}(y) = \inf_{\hat{y} \geq y} G_d(\hat{y})$ that is (i') a nondecreasing function and (ii') satisfying $G_d(y) \geq G_d^{\text{inf}}(y)$ for all $y \in \mathbb{Y}$. It further holds that $G_d^{\text{inf}}(y)$ is such that $G_d^{\text{inf}}(y) \geq \tilde{G}_d(y)$ for all $y \in \mathbb{Y}$, where \tilde{G}_d is any real-valued function \tilde{G}_d satisfying conditions (i') and (ii').

Part I. Using (5) and that distribution functions are nondecreasing, any feasible candidate of $F_{Y_d^{CO}}$ has to satisfy that, for any $y, y' \in \mathbb{Y}$ and $y' \leq y$,

$$F_{Y_d^{CO}}(y) - F_{Y_d^{CO}}(y') \leq \frac{Q_{dd}(y) - Q_{dd}(y')}{\pi_{CO}}. \quad (30)$$

Using the same reasoning and (6), it follows that

$$F_{Y_d^{CO}}(y) - F_{Y_d^{CO}}(y') \geq \frac{\pi_{\Delta}}{\pi_{CO}} (G_d(y) - G_d(y')).$$

for any arbitrary y and y' . As $G_d(y)$ is not necessarily nondecreasing, we can similarly conclude that it has to hold that

$$\mathbb{P}(Y_d^{CO} \in B) \geq \frac{\pi_{\Delta}}{\pi_{CO}} (\mathbb{P}(Y \in B, D = d | Z = d) - \mathbb{P}(Y \in B, D = d | Z = 1 - d))$$

for any $B \in \mathcal{B}$ and therefore

$$F_{Y_d^{CO}}(y) - F_{Y_d^{CO}}(y') \geq \frac{\pi_{\Delta}}{\pi_{CO}} (G_d^+(y) - G_d^+(y')). \quad (31)$$

Any feasible candidate of $F_{Y_d^{CO}}$ has to further satisfy the conditions

$$(viii) \quad \lim_{y \rightarrow \underline{y}} F_{Y_d^{CO}}(y) = 0 \quad \text{and} \quad \lim_{y \rightarrow \bar{y}} F_{Y_d^{CO}}(y) = 1. \quad (32)$$

The distribution functions $F_{Y_d^{DF}}$, and $F_{Y_d^{dT}}$ fulfill then these limit conditions based on (5) and (6), as it holds that $\lim_{y \rightarrow \underline{y}} G_d(y) = \lim_{y \rightarrow \underline{y}} Q_{ds}(y) = 0$ and $\lim_{y \rightarrow \bar{y}} G_d(y) = 1$ and $\lim_{y \rightarrow \bar{y}} Q_{dd}(y) = \pi_d + \pi_{CO}$ and $\lim_{y \rightarrow \bar{y}} Q_{d(1-d)}(y) = \pi_d + \pi_{DF}$ for any $d, s \in \{0, 1\}$.

Any real-valued function, which is defined on \mathbb{Y} and right-continuous, which left-limits exists and which satisfy equations (29) – (32) implies by construction potential outcome distributions for all four groups, which are consistent with the imposed model assumption, the sensitivity parameter constraints, and the observed probability functions. It is thus a feasible candidate of $F_{Y_d^{CO}}$. It is clear that the simple additive structure of all imposed conditions implies that if there are two different such feasible candidate functions, any convex combinations of these functions satisfy these conditions as well.

Part II. We show in the following both that the proposed bounds $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$ satisfy the constraints in (29) – (32) and that any function which takes values outside of these bounds contradicts one of these conditions and is therefore incompatible with the distribution of (Y, D, Z) , our assumption and the sensitivity parameters. As the considered sensitivity parameters lie within

the sensitivity region by assumption, bounds on the outcome distribution of compliers exist and are therefore non-intersecting by construction. The condition in (29) is therefore satisfied if our bounds $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$ satisfy

$$\underline{H}_{Y_d^{CO}}(y) \leq \underline{F}_{Y_d^{CO}}(y) \quad \text{and} \quad \overline{F}_{Y_d^{CO}}(y) \leq \overline{H}_{Y_d^{CO}}(y) \quad (33)$$

for all $y \in \mathbb{Y}$. Additionally, both bounds preserve the existence of limits and continuity.

Part II - Lower Bound. We consider first

$$\underline{H}_1(y) = \frac{1}{\pi_{CO}} \left(\pi_{\Delta} G_d^+(y) - \inf_{\tilde{y} \leq y} \left(\pi_{\Delta} G_d^+(\tilde{y}) - \pi_{CO} \underline{H}(\tilde{y}) \right) \right). \quad (34)$$

It clearly holds that $\underline{H}_1(y) \geq \underline{H}_{Y_d^{CO}}(y)$. Consider again any $y, y' \in \mathbb{Y}$ such that $y' \leq y$. Based on this reasoning $\underline{H}_1(y)$ satisfies constraint (31) as

$$\begin{aligned} & \underline{H}_1(y) - \underline{H}_1(y') \\ &= \frac{1}{\pi_{CO}} \left(\pi_{\Delta} G_d^+(y) - \pi_{\Delta} G_d^+(y') - \inf_{\tilde{y} \leq y} \left(\pi_{\Delta} G_d^+(\tilde{y}) - \underline{H}(\tilde{y}) \right) - \inf_{\tilde{y} \leq y'} \left(\pi_{\Delta} G_d^+(\tilde{y}) - \underline{H}_{Y_d^{CO}}(\tilde{y}) \right) \right) \\ &\geq \frac{1}{\pi_{CO}} \left(\pi_{\Delta} G_d^+(y) - \pi_{\Delta} G_d^+(y') \right). \end{aligned}$$

Any function F such that $F(y) \leq \underline{H}_1(y)$ either violates (31) or (33) for some $y \in \mathbb{Y}$. We conclude that any feasible candidate function of $F_{Y_d^{CO}}$ has to satisfy

$$\underline{H}_1(y) \leq F_{Y_d^{CO}}(y) \quad (35)$$

We now consider our final lower bound

$$\underline{F}_{Y_d^{CO}}(y) = \frac{1}{\pi_{CO}} \left(Q_{dd}(y) - \inf_{\tilde{y} \geq y} \left(Q_{dd}(\tilde{y}) - \underline{H}_1(\tilde{y}) \right) \right).$$

It is clear that $\underline{F}_{Y_d^{CO}}(y) \geq \underline{H}_1(y)$ and that $Q_{dd}(y) - Q_{dd}(y') \geq \underline{F}_{Y_d^{CO}}(y) - \underline{F}_{Y_d^{CO}}(y')$. As it further holds that, for any $y, y' \in \mathbb{Y}$ and $y' \leq y$,

$$Q_{dd}(y) - Q_{dd}(y') \geq G_d^+(y) - G_d^+(y')$$

$\underline{F}_{Y_d^{CO}}(y)$ satiates (31), (30), and it holds that $\underline{F}_{Y_d^{CO}}(y) \geq \underline{H}_1(y)$. Clearly, any function F such that $F(y) \leq \underline{H}_1(y)$ is incompatible with the distribution of (Y, D, Z) , the sensitivity parameters and our assumptions.

We now show that $\underline{F}_{Y_d^{CO}}(y)$ satisfies (32). By construction, $\underline{F}_{Y_d^{CO}} \in [0, 1]$. We therefore show

that $\lim_{y \rightarrow \underline{y}} \underline{F}_{Y_d^{CO}}(y) \leq 0$ and $\lim_{y \rightarrow \bar{y}} \underline{F}_{Y_d^{CO}}(y) \geq 1$. It holds that

$$\lim_{y \rightarrow \underline{y}} \underline{F}_{Y_d^{CO}}(y) = \frac{1}{\pi_{CO}} \inf_{\tilde{y} \in \mathbb{R}} \left(Q_{dd}(\tilde{y}) - (\pi_{\Delta} G_d^+(\tilde{y}) - \inf_{\hat{y} \leq \tilde{y}} (\pi_{\Delta} G_d^+(\hat{y}) + \pi_{CO} \underline{H}_{Y_d^{CO}}(\hat{y}))) \right)$$

The equality follows as $\lim_{y \rightarrow \underline{y}} Q_{dd}(y) = 0$. We note that for all $y, y' \in \mathbb{Y}$ and $y' \leq y$ $Q_{dd}(y) - Q_{dd}(y') \geq \pi_{\Delta} (G_d^+(y) - G_d^+(y'))$. It follows that

$$\begin{aligned} & \lim_{y \rightarrow \underline{y}} \underline{F}_{Y_d^{CO}}(y) \\ & \leq \frac{1}{\pi_{CO}} \inf_{\hat{y} \in \mathbb{Y}} \left(\underbrace{\max\{0, Q_{dd}(\hat{y}) - \pi_d, \pi_{\Delta} G_d^{\text{sup}}(\hat{y})\}}_{(1a)} \underbrace{- \pi_d}_{(2a)} \underbrace{+ \pi_{\Delta} G_d^{\text{sup}}(\hat{y})}_{(3a)} \underbrace{- \pi_{CO} \frac{\pi_{DF}}{\pi_{\Delta}} \delta}_{(4a)} \right) - Q_{dd}(\hat{y}). \end{aligned}$$

We now show that each of the expressions (1a)–(4a) evaluated at any $\hat{y} \in \mathbb{Y}$ is bounded by $Q_{dd}(\hat{y})$ so that it holds that $\lim_{y \rightarrow \underline{y}} \pi_{CO} \underline{F}_{Y_d^{CO}}(y) \leq 0$. It is obvious that expressions (1a) and (2a) satisfy this reasoning. Considering (3a), we note that it holds that $\pi_{\Delta} G_d^{\text{sup}}(\hat{y}) \leq Q_{dd}(\hat{y})$. We turn to (4a). It holds that

$$G_d^{\text{sup}}(\hat{y}) - \frac{\pi_{DF}}{\pi_{\Delta}} \delta - \frac{Q_{dd}(\hat{y})}{\pi_{CO}} \leq F_{Y_d^{CO}}(\hat{y}) + \frac{\pi_{DF}}{\pi_{\Delta}} \delta - \frac{\pi_{DF}}{\pi_{\Delta}} \delta - \frac{Q_{dd}(\hat{y})}{\pi_{CO}} \leq F_{Y_d^{CO}}(\hat{y}) - \frac{Q_{dd}(\hat{y})}{\pi_{CO}} \leq 0$$

We consider the right limit. It holds that $\underline{F}_{Y_d^{CO}} \geq \underline{H}_{Y_d^{CO}}$ and therefore

$$\lim_{y \rightarrow \bar{y}} \underline{F}_{Y_d^{CO}}(y) \geq \lim_{y \rightarrow \bar{y}} \max\{0, G_d^{\text{sup}}(y) - \frac{\pi_{DF}}{\pi_{\Delta}} \delta, \frac{\pi_{\Delta}}{\pi_{CO}} G_d^{\text{sup}}(y), \frac{Q_{dd}(y) - \pi_d}{\pi_{CO}}\} \geq 1.$$

The second inequality follows as $\lim_{y \rightarrow \bar{y}} Q_{dd}(y) - \pi_d = \pi_{CO} + \pi_d - \pi_d = \pi_{CO}$. This reasoning concludes the proof of the lower bound.

Part II - Upper Bound. A similar reasoning applies to the upper bound. To briefly sketch this reasoning, let

$$\bar{H}_1(y) = Q_{dd}(y) - \sup_{\hat{y} \leq y} \left(Q_{dd}(\hat{y}) - \pi_{CO} \bar{H}_{Y_d^{CO}}(\hat{y}) \right).$$

It is clear that $\bar{H}_1(y) \geq \bar{H}_{Y_d^{CO}}(y)$ and that $Q_{dd}(y) - Q_{dd}(y') \geq \bar{H}_1(y) - \bar{H}_1(y')$. It holds that $\bar{H}_1(y')$ satisfies (30). Clearly, any function F such that $F(y) \leq \bar{H}_1(y)$ is incompatible with the distribution of (Y, D, Z) , the sensitivity parameters and our assumptions. It therefore follows that any function which is a feasible candidate of the distribution function $F_{Y_d^{CO}}$ has to satisfy

$$F_{Y_d^{CO}}(y) \leq \bar{H}_1(y) \tag{36}$$

We now consider our proposed bound.

$$\bar{F}_{Y_d^{CO}}(y) = \pi_{\Delta} G_d^+(y) - \sup_{\tilde{y} \geq y} \left(\pi_{\Delta} G_d^+(\tilde{y}) - \bar{H}_1(y) \right).$$

It follows from the same reasoning as above that $\bar{F}_{Y_d^{CO}}(y)$ satisfies (36), (30) and (31). Clearly, any function F such that $F(y) \geq \bar{F}_{Y_d^{CO}}(y)$ is incompatible with the distribution of (Y, D, Z) , the sensitivity parameters and our assumptions.

We conclude by showing that $\bar{F}_{Y_d^{CO}}(y)$ satisfies (32). It holds that

$$\lim_{y \rightarrow \underline{y}} \bar{F}_{Y_d^{CO}}(y) \leq \lim_{y \rightarrow \underline{y}} \min \left\{ 1, G_d^{\text{inf}}(y) + \frac{\pi_{\text{DF}}}{\pi_{\Delta}} \delta, \frac{\pi_{\text{CO}}}{\pi_{\Delta}} G_d^{\text{inf}}(y) + \frac{\pi_{\text{CO}}}{\pi_{\text{DF}}}, \frac{Q_{dd}(y)}{\pi_{\text{CO}}} \right\} \leq 0,$$

where the second inequality follows by $\lim_{y \rightarrow \underline{y}} \frac{Q_{dd}(y)}{\pi_{\text{CO}}} = 0$. We now consider

$$\begin{aligned} & \lim_{y \rightarrow \bar{y}} \bar{F}_{Y_d^{CO}}(y) \\ &= \frac{\pi_{\text{CO}} + \pi_d}{\pi_{\text{CO}}} - \sup_{\hat{y} \in Y} \left(\frac{Q_{dd}(\hat{y})}{\pi_{\text{CO}}} - \min \left\{ \underbrace{1}_{(1b)}, \underbrace{\frac{Q_{dd}(\hat{y})}{\pi_{\text{CO}}}}_{(2b)}, \underbrace{\frac{\pi_{\Delta}}{\pi_{\text{CO}}} G_d^{\text{inf}}(\hat{y}) + \frac{\pi_{\text{DF}}}{\pi_{\text{CO}}}}_{(3b)}, \underbrace{G_d^{\text{inf}}(\hat{y}) + \frac{\pi_{\text{DF}}}{\pi_{\Delta}} \delta}_{(4b)} \right\} \right). \end{aligned}$$

We show that (1b)–(4b) are bounded from below by $\frac{Q_{dd}(\hat{y})}{\pi_{\text{CO}}} - \frac{\pi_d}{\pi_{\text{CO}}}$ such that

$$\lim_{y \rightarrow \bar{y}} \bar{F}_{Y_d^{CO}}(y) \geq \frac{\pi_{\text{CO}} + \pi_d}{\pi_{\text{CO}}} - \frac{\pi_d}{\pi_{\text{CO}}} = 1.$$

It is clear that (1b)–(2b) satisfies this restriction. Concerning (3b), we note that

$$\frac{1}{\pi_{\text{CO}}} \left(\pi_{\Delta} G_d^{\text{inf}}(\hat{y}) + \pi_{\text{DF}} \right) \geq \frac{1}{\pi_{\text{CO}}} \left(Q_{dd}(\hat{y}) + \pi_{\text{DF}} - \pi_d - \pi_{\text{DF}} \right) = \frac{Q_{dd}(\hat{y}) - \pi_d}{\pi_{\text{CO}}}.$$

Concerning (4b), we note that

$$G_d^{\text{inf}}(\hat{y}) + \frac{\pi_{\text{DF}}}{\pi_{\Delta}} \delta \geq F_{Y_d^{CO}}(\hat{y}) - \frac{\pi_{\text{DF}}}{\pi_{\Delta}} \delta + \frac{\pi_{\text{DF}}}{\pi_{\Delta}} \delta \geq \frac{Q_{dd}(\hat{y})}{\pi_{\text{CO}}} - \frac{\pi_d}{\pi_{\text{CO}}}$$

This completes this proof. □

B.2. Proof of Proposition 1. We show that the population size of compliers is sharply bounded by $\underline{\pi}_{CO} \leq \pi_{CO} \leq \bar{\pi}_{CO}$, where for $B \in \mathcal{B}$ and $d, z \in \{0, 1\}$

$$\begin{aligned}\bar{\pi}_{CO} &= \min\{\mathbb{P}(D = 1|Z = 1), \mathbb{P}(D = 0|Z = 0)\} \\ \underline{\pi}_{CO} &= \max_{d \in \{0,1\}} \left\{ \sup_{B \in \mathcal{B}} \{\mathbb{P}(Y \in B, D = d|Z = d) - \mathbb{P}(Y \in B, D = d|Z = 1 - d)\} \right\}.\end{aligned}$$

The proposition follows from this statement as $\pi_{DF} = \pi_{CO} - \mathbb{P}(D = 1|Z = 1) + \mathbb{P}(D = 1|Z = 0)$. Let $\mathbb{P}(Y_d^t \in B)$ denotes the unobserved probability distribution of the potential outcome of group t with treatment status d .³⁴

$\underline{\pi}_{CO}$ is a valid lower bound of the population size of compliers as it follows from the definition of groups that

$$\bar{\pi}_{CO} = \min\{\pi_{AT} + \pi_{CO}, \pi_{NT} + \pi_{CO}\} \geq \pi_{CO}.$$

Similarly, $\underline{\pi}_{CO}$ bounds the population size of compliers from below as $\underline{\pi}_{CO}$ equals

$$\begin{aligned}\max & \left\{ \sup_{B \in \mathcal{B}} \{\mathbb{P}(Y_1 \in B, AT) + \mathbb{P}(Y_1 \in B, CO) - \mathbb{P}(Y_1 \in B, AT) - \mathbb{P}(Y_1 \in B, DF)\}, \right. \\ & \left. \sup_{B \in \mathcal{B}} \{\mathbb{P}(Y_0 \in B, NT) + \mathbb{P}(Y_0 \in B, CO) - \mathbb{P}(Y_0 \in B, NT) - \mathbb{P}(Y_0 \in B, DF)\} \right\} \\ & \leq \max\left\{ \sup_{B \in \mathcal{B}} \{\mathbb{P}(Y_1 \in B, CO)\}, \sup_{B \in \mathcal{B}} \{\mathbb{P}(Y_0 \in B, CO)\} \right\} = \pi_{CO}.\end{aligned}$$

The inequality follows from the independence assumption and the definition of the groups. It is therefore clear that the population size of compliers lies within the bounds. It remains to show that these bounds are sharp. To show this, we consider any fix $\tilde{\pi}_{CO} \in [\underline{\pi}_{CO}, \bar{\pi}_{CO}]$. Let $B \in \mathcal{B}$ and $\mathcal{B}_B = \{A \cap B | A \in \mathcal{B}\}$. Using the discussion of the proof of Theorem 1 about how to verify that a candidate distribution is a feasible distribution, we consider the following marginal outcome

³⁴In principle, Proposition 1 is a Corollary of Theorem 1. Considering the sharp lower bound on the population size of defiers, one could simply use the bounds to solve for the minimal size of defiers for which there exists one value of outcome heterogeneity δ such that the bounds are non-intersecting. However, this exercise is tedious, and we propose a simpler and direct proof for this claim in this section.

distributions of the groups.³⁵

$$\begin{aligned}\mathbb{P}(\tilde{Y}_d \in B, T = CO) &= \mathbb{P}(Y \in B, D = d|Z = d) - \mathbb{P}(\tilde{Y}_d \in B, T = dT), \\ \mathbb{P}(\tilde{Y}_d \in B, T = DF) &= \mathbb{P}(Y \in B, D = d|Z = 1 - d) - \mathbb{P}(\tilde{Y}_d \in B, T = dT), \\ \mathbb{P}(\tilde{Y}_d \in B, T = dT) &= L_1 \cdot L_2,\end{aligned}$$

where

$$\begin{aligned}L_1 &= \frac{\mathbb{P}(D = d|Z = d) - \tilde{\pi}_{CO}}{\mathbb{P}(D = d|Z = d) - \sup_{C \in \mathcal{B}}(\mathbb{P}(Y \in C, D = d|Z = d) - \mathbb{P}(Y \in C, D = d|Z = 1 - d))}, \\ L_2 &= \mathbb{P}(Y \in B, D = d|Z = d) \\ &\quad - \sup_{C \in \mathcal{B}_B} (\mathbb{P}(Y \in C, D = d|Z = d) - \mathbb{P}(Y \in C, D = d|Z = 1 - d)).\end{aligned}$$

The outcome distribution of group dT is the product of two terms. The term L_1 guarantees that the probability distributions integrate to the corresponding population size. The term L_2 guarantees that the outcome probabilities of compliers and defiers are nonnegative. The outcome distributions of the other groups are respectively defined.

By construction, the proposed outcome probability distributions imply the observed outcome probability distributions.³⁶ We show now that the implied probability distributions are indeed distributions, which satisfy $\forall T \in \{CO, DF, AT, NT\}$, $d \in \{0, 1\}$, and $B, B' \in \mathcal{B}$, where $B' \subseteq B$: (i) $\mathbb{P}(\tilde{Y}_d^T \in \mathbb{Y}) = 1$; (ii) $\mathbb{P}(\tilde{Y}_d^T \in B) \geq 0$; and (iii) $\mathbb{P}(\tilde{Y}_d^T \in B) \geq \mathbb{P}(\tilde{Y}_d^T \in B')$. We consider any $B \in \mathcal{B}$ and any $d \in \{0, 1\}$ in the following.

We first consider condition (i). It clearly holds that $\mathbb{P}(\tilde{Y}_d \in \mathbb{Y}, T = CO) = \tilde{\pi}_{CO}$, and

$$\begin{aligned}\mathbb{P}(\tilde{Y}_d \in \mathbb{Y}, T = dT) &= \mathbb{P}(D = d|Z = d) - \tilde{\pi}_{CO} = \tilde{\pi}_{dT} \\ \mathbb{P}(\tilde{Y}_d \in \mathbb{Y}, T = DF) &= \mathbb{P}(D = d|Z = 1 - d) - \mathbb{P}(D = d|Z = d) + \tilde{\pi}_{CO} = \tilde{\pi}_{DF}.\end{aligned}$$

We turn to condition (ii). First note that it follows from the bounds on the population size of compliers $\underline{\pi}_{CO}$ that $0 \leq L_1 \leq 1$. Second, we note that

$$L_2 \geq \mathbb{P}(Y \in B, D = d|Z = d) - \sup_{C \in \mathcal{B}_B} (\mathbb{P}(Y \in C, D = d|Z = d)) = 0.$$

³⁵Otherwise $\mathbb{P}(\tilde{Y}_d \in B, T = dT)$ is defined to be zero if $\mathbb{P}(D = d|Z = d) = \sup_{C \in \mathcal{B}}(\mathbb{P}(Y \in C, D = d|Z = d) - \mathbb{P}(Y \in C, D = d|Z = 1 - d))$. The other probability distributions stay the same.

³⁶This means that $\forall B \in \mathcal{B}$ and $\forall d, z \in \{0, 1\}$ $\mathbb{P}(Y \in B, D = d|Z = z) = \mathbb{P}(\tilde{Y} \in B, \tilde{D} = d|\tilde{Z} = z)$.

This reasoning implies that $\mathbb{P}(\tilde{Y}_d \in B, T = dT) \geq 0$. Further, note that

$$\begin{aligned} \mathbb{P}(\tilde{Y}_d \in B, T = DF) &= \mathbb{P}(Y \in B, D = d|Z = 1 - d) - \mathbb{P}(\tilde{Y}_d \in B, T = dT) \\ &= \mathbb{P}(Y \in B, D = d|Z = 1 - d) - L_1 \mathbb{P}(Y \in B, D = d|Z = d) \\ &\quad + L_1 \sup_{C \in \mathcal{B}_B} (\mathbb{P}(Y \in C, D = d|Z = d) - \mathbb{P}(Y \in C, D = d|Z = 1 - d)) \geq 0, \end{aligned}$$

by basic arguments about sets. A similar reasoning applies to the compliers.

We consider condition (iii). Let $B' \subseteq B$. We note that

$$\begin{aligned} &\mathbb{P}(\tilde{Y}_d \in B, T = dT) - \mathbb{P}(\tilde{Y}_d \in B', T = dT) \\ &\geq \mathbb{P}(Y \in B \setminus B', D = d|Z = d) \\ &\quad - \sup_{C \in \mathcal{B}_{B \setminus B'}} (\mathbb{P}(Y \in C, D = d|Z = d) - \mathbb{P}(Y \in C, D = d|Z = 1 - d)) \geq 0. \end{aligned}$$

Using a simple arguments, it further holds that $\mathbb{P}(\tilde{Y}_d \in B, T = DF) \geq \mathbb{P}(\tilde{Y}_d \in B', T = DF)$ as

$$\begin{aligned} &\mathbb{P}(\tilde{Y}_d \in B, T = dT) - \mathbb{P}(\tilde{Y}_d \in B', T = dT) \\ &= \mathbb{P}(Y \in B, D = d|Z = d) - \mathbb{P}(Y \in B', D = d|Z = d) \\ &\quad - \sup_{C \in \mathcal{B}_{B \setminus B'}} (\mathbb{P}(Y \in C, D = d|Z = d) - \mathbb{P}(Y \in C, D = d|Z = 1 - d)) \\ &\leq \mathbb{P}(Y \in B, D = d|Z = 1 - d) - \mathbb{P}(Y \in B', D = d|Z = 1 - d). \end{aligned}$$

A similar reasoning applies to the compliers, which completes this proof. \square

B.3. Proof of Proposition 2. It holds by our assumptions that, for $d \in \{0, 1\}$,

$$F_{Y_d}(y) = \pi_{1-d} F_{Y_d^{(1-d)T}}(y) + \pi_{CO} F_{Y_d^{CO}}(y) + \pi_{DF} F_{Y_d^{DF}}(y) + \pi_d F_{Y_d^{dT}}(y),$$

where π_{1-d} is the population size of always takers if $d = 0$ and otherwise of never takers and $F_{Y_d^{(1-d)T}}$ respectively. In the absence of treatment, the data generating process does not reveal anything about the distribution of the always takers, and neither in the presence of treatment of the never takers. The proof of Theorem 1 implies sharp bounds on the remaining six potential outcome

distributions. Using (5) and (6), it follows that

$$\begin{aligned}
F_{Y_d}(y) &= \pi_{1-d}F_{Y_d^{(1-d)T}}(y) + \pi_{CO}F_{Y_d^{CO}}(y) - \pi_{\Delta}G_d(y) + \pi_{CO}F_{Y_d^{CO}}(y) + Q_{dd}(y) - \pi_{CO}F_{Y_d^{CO}}(y). \\
&= \pi_{1-d}F_{Y_d^{(1-d)T}}(y) + \pi_{CO}F_{Y_d^{CO}}(y) - \pi_{\Delta}G_d(y) + Q_{dd}(y) \\
&= \pi_{1-d}F_{Y_d^{(1-d)T}}(y) + \pi_{CO}F_{Y_d^{CO}}(y) + Q_{d(1-d)}(y)
\end{aligned}$$

Sharp bounds in a first-order stochastic dominance sense of $F_{Y_d}(y)$ are therefore obtained by Theorem 1 by taking the distribution functions $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$ and setting $F_{Y_d^{(1-d)T}}(y)$ to its most extreme values, respectively. The statement follows from this reasoning. \square

B.4. Proof of Corollary 1. The statement follows directly from first-order stochastic dominance of the distribution functions $\underline{F}_{Y_d^{CO}}$ and $\overline{F}_{Y_d^{CO}}$ by Theorem 1 and Lemma 1 in Stoye (2010). \square

B.5. Proof of Corollary 2. The statement directly follows from Theorem 1 by noting how the bounds simplify for a binary variable. \square

B.6. Proof of Proposition A.1. By the same arguments of the proof of Theorem 1, one can show that these bounds are sharp conditionally on the covariates given the respective assumptions. \square

B.7. Verification of Expressions used throughout the Paper. In this section, we verify a few simple expressions that we use through the text for completeness.

B.7.1. Verification of Properties of the function $G_d^+(y)$.

Lemma B.1. *Suppose $Q_{ds}(y)$ is continuously differentiable in $y \in \mathbb{Y}$ for $d, s \in \{0, 1\}$. Then,*

$$G_d^+(y) = \int_{\mathbb{Y}} \mathbf{1}\{z \leq y\} \max\{0, g_d(z)\} dz. \quad (37)$$

Proof: We note that

$$\begin{aligned}
G_d^+(y) &= \frac{1}{\pi_{\Delta}} \sup_{B \in \mathcal{B}} \{\mathbb{P}(Y \in B, Y \leq y, D = d | Z = d) - \mathbb{P}(Y \in B, Y \leq y, D = d | Z = 1 - d)\} \\
&= \frac{1}{\pi_{\Delta}} \sup_{B \in \mathcal{B}} \left\{ \int_{\mathbb{Y}} \mathbf{1}\{z \in B\} \mathbf{1}\{z \leq y\} q_{dd}(z) dz - \int_{\mathbb{Y}} \mathbf{1}\{z \in B\} \mathbf{1}\{z \leq y\} q_{d(1-d)}(z) dz \right\} \\
&= \int_{\mathbb{Y}} \mathbf{1}\{z \leq y\} \max\{0, g_d(z)\} dz.
\end{aligned}$$

The first inequality follows from the definition of probabilities and our definition of $q_{ds}(z)$. The second equality follows by continuity of $q_{d(1-d)}$. \square

B.7.2. *Outer and Inner Set for Sensitivity and Robust Region.* We first verify that our expression (26) follows from expression (27). Let it holds that $\phi_L(\pi_{\text{DF}}; \theta) \leq \phi(\pi_{\text{DF}}; \theta)$ for each component and for all $\pi_{\text{DF}} \in [0, 0.5)$. We denote the l -th unit vector by e_l . We then note that by the definition of $\phi_L(\pi_{\text{DF}}; \theta)$ and SR of Section 5 that

$$\begin{aligned} SR &= \left\{ (\pi_{\text{DF}}, \delta) : e_1^\top \phi(\theta, \pi_{\text{DF}}) \leq \pi_{\text{DF}} \leq -e_2^\top \phi(\theta, \pi_{\text{DF}}) \right. \\ &\quad \left. e_3^\top \phi(\theta, \pi_{\text{DF}}) \leq \delta \leq -e_4^\top \phi(\theta, \pi_{\text{DF}}) \right\} \\ &\subseteq \left\{ (\pi_{\text{DF}}, \delta) : e_1^\top \phi_L(\theta, \pi_{\text{DF}}) \leq \pi_{\text{DF}} \leq -e_2^\top \phi_L(\theta, \pi_{\text{DF}}) \right. \\ &\quad \left. e_3^\top \phi_L(\theta, \pi_{\text{DF}}) \leq \delta \leq -e_4^\top \phi_L(\theta, \pi_{\text{DF}}) \right\} = SR_L. \end{aligned}$$

By a similar argument we note that

$$\begin{aligned} RR_L(SR_L) &= \{(\pi_{\text{DF}}, \delta) \in SR_L : \delta \leq e_5^\top \phi_L(\theta, \pi_{\text{DF}})\} \\ &\supseteq \{(\pi_{\text{DF}}, \delta) \in SR_L : \delta \leq e_5^\top \phi(\theta, \pi_{\text{DF}})\} = RR(SR_L). \end{aligned}$$

As we have shown above that $SR \subseteq SR_L$ it follows that $RR_L(SR) \supseteq RR(SR)$. \square

C. ADDITIONAL MATERIALS FOR ESTIMATION AND INFERENCE

In this section, we present more details on the estimation and inference methods proposed in the main text. We first consider a binary and then a continuous outcome variable. For both cases, we provide more details about estimating the sensitivity and robust regions, we discuss the imposed assumptions and then proceed by showing asymptotic results. Many of the following results are based on applications and ideas from other papers and we therefore only sketch most of them.

C.1. Estimation for a binary outcome variable. We consider a binary outcome variable, where the mapping of interest is $\phi_b(\theta_b, \pi_{\text{DF}})$. As shown in Section 6.2, the underlying parameters θ_b are given by $(P_{11}, P_{10}, P_{01}, P_{00}, P_0, P_1)$. We estimate the probabilities by their sample counterparts, i.e. $P_{ds} = \frac{1}{n_s} \sum_{i=1}^{n_s} \mathbf{1}\{Y_i^s = 1, D_i^s = d\}$ and $P_s = \frac{1}{n_s} \sum_{i=1}^{n_s} \mathbf{1}\{D_i^s = 1\}$. We then estimate $\phi_b(\hat{\theta}_b, \pi_{\text{DF}})$ by simple plug-in estimates, where the precise formulas are given in Section 6.2 and Appendix A.4.

C.2. Assumptions. We consider the following sampling process.

Assumption C.1. For $z \in \{0, 1\}$, $\{(Y_i^z, D_i^z)\}_{i=1}^{n_z}$ are identically and independently distributed according to the distribution of (Y^z, D^z) which is drawn conditional on $Z = z$ with support $\mathbb{Y} \times \{0, 1\}$. It holds that n_0/n converges to a nonzero constant as $n \rightarrow \infty$.

By Assumption 1, the instrument is independent of all potential outcomes, so that the distribution

of the instrument does not contain any further information and we can assume that the sampling is conditionally on the instrument (see, e.g., Kitagawa, 2015).

C.3. Inference for a binary outcome variable. In this section, we present more details on how to construct confidence sets for a binary outcome variable. Based on the derivation in Section 6.2 and in Appendix A.4, it suffices to construct a lower confidence band for $\tilde{\phi}_b(\hat{\theta}_b, \pi_{\text{DF}})$ given in (28). To unify the notation, let us denote the i -th component of this mapping by $\phi_{b,i}(\theta_b, \pi_{\text{DF}})$ for $i \in \{1, \dots, 6\}$. We note that each of these components can be written as

$$\phi_{b,i}(\theta_b, \pi_{\text{DF}}) = \max_{j=1}^{J(i)} \{\psi_{i,j}(\theta_b, \pi_{\text{DF}})\},$$

where $\psi_{i,j}(\theta_b, \pi_{\text{DF}})$ are Hadamard-differentiable functions of $(\theta_b, \pi_{\text{DF}})$ by the relevance assumption. The mappings $\phi_{b,i}(\theta_b, \pi_{\text{DF}})$ are not Hadamard-differentiable on $(\theta_b, \pi_{\text{DF}})$, but they are Hadamard-directionally-differentiable in the direction of θ_b when evaluated at any finite set of $\{\pi_{\text{DF}}^k\}_{k=1}^K$, where $\pi_{\text{DF}}^k \in [0, 0.5]$ and K is some finite number.

Following ideas of Fang and Santos (2018) and Masten and Poirier (2020), we consider a bootstrap method to construct confidence sets $\tilde{\phi}_{b,i}(\theta_b, \pi_{\text{DF}}^k)$ which are uniformly valid across k and i . Specifically, the directional derivative of $\tilde{\phi}_{b,i}(\theta_b, \pi_{\text{DF}})$ in the direction of θ_b evaluated at some π_{DF} is given by

$$\tilde{\phi}'_{i,b,\theta_b}(h, \pi_{\text{DF}}) = \max_{j:\psi_{1,j}(\theta_b, \pi_{\text{DF}}) \geq \max_{s \leq J(i)} \{\psi_{1,s}(\theta_b, \pi_{\text{DF}})\}} h_j,$$

for all $h \in \mathbb{R}^{J(i)}$.³⁷ Following Fang and Santos (2018), we consider as an estimator of this directional derivative,

$$\tilde{\phi}'_{i,b,\theta_b}(h, \pi_{\text{DF}}) = \max_{j:\psi_{1,j}(\theta_b, \pi_{\text{DF}}) \geq \max_{s \leq J(i)} \{\psi_{1,s}(\theta_b, \pi_{\text{DF}}) + \kappa\}} h_j,$$

where $\kappa > 0$ and $\kappa \rightarrow 0$ and $\kappa\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$.

We first get estimates of θ_b and $\phi_b(\theta_b, \pi_{\text{DF}}^k)$ from the original sample for all $k \in \{1, \dots, K\}$. We then generate B bootstrap samples $\{(Y_i^{b,z}, D_i^{b,z})\}_{i=1}^{n_z}$, $b = 1, \dots, B$ by drawing n_z observations with replacements from the original data $\{Y_i^z, D_i^z\}_{i=1}^{n_z}$ for $z \in \{0, 1\}$ and we calculate $\hat{\phi}'_{b,\theta}$ for each bootstrap iteration. We take

$$\widehat{\text{cv}}_{1-\alpha} = \inf(z : \mathbb{P}(\max_{k \in \{1, \dots, K\}} \tilde{\phi}'_{b,\theta}((\sqrt{n}(\hat{\theta}_b^* - \hat{\theta}_b); \pi_{\text{DF}}^k) - z) \leq 0) \geq 1 - \alpha'),$$

where $\alpha' < \alpha$ but arbitrarily close to α .³⁸ We then consider as lower confidence set $\tilde{\phi}_b(\hat{\theta}_b, \pi_{\text{DF}}^k) -$

³⁷See Definition 2.1 in Fang and Santos (2018) for a definition of Hadamard-directionally differentiable mappings.

³⁸To simplify the notation, we just consider a fix critical value $\widehat{\text{cv}}_{1-\alpha}$ here. In principle, it might be different for each component and for each point of evaluation $\{\pi_{\text{DF}}^k\}_{k=1}^K$ and indeed it would be more efficient to do this.

$\widehat{c\mathbf{v}}_{1-\alpha}/\sqrt{n}$ for all $k \in \{1, \dots, K\}$. These lower confidence sets are uniformly valid for the mapping $\tilde{\phi}_b$ when evaluated at $\{\pi_{\text{DF}}^k\}_{k=1}^K$.

To obtain a lower confidence band of $\tilde{\phi}_b$, which is valid uniformly in π_{DF} , we exploit the functional form of $\tilde{\phi}_b$ similarly to Masten and Poirier (2020). The lower bound for intermediates points, that are not within the set $\{\pi_{\text{DF}}^k\}_{k=1}^K$, is interpolated based on the left and right nearest neighbor of the point of evaluation. The respectively lowest confidence set is taken. By monotonicity of $\tilde{\phi}_b$, this lower confidence set is then also valid uniformly in π_{DF} .

To construct a valid confidence set for ϕ_b , we then consider a simple projection argument of $\tilde{\phi}_b$ by taking the maximum of the last three components of $\tilde{\phi}_b$ into account. We construct our confidence set for our sensitivity and our robust region $\widehat{RR}_{b,L}$ and $\widehat{SR}_{b,L}$ based on our constructed lower confidence set.

Proposition C.1. *Suppose that Assumption 1 and C.1 hold and the variance of each component of θ_b is bounded away from zero. It then holds that,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\widehat{RR}_{b,L} \subseteq RR_b, \widehat{SR}_b \subseteq \widehat{SR}_{b,L}) \geq 1 - \alpha.$$

We impose the variance condition to ensure that the underlying parameters converge to a non-degenerated distribution.

C.4. Estimation for a Continuous Outcome Variable. In this section, we give further details on the construction of the estimators for a continuous outcome variable. We first estimate the underlying parameters θ . We estimate the conditional joint densities by standard nonparametric kernel density estimator

$$\widehat{q}_{dz}(y) = \frac{1}{n_z h} \sum_{i=1}^{n_z} K_h(Y_i^z - y) \cdot \mathbf{1}\{D_i^z = d\},$$

where $K_h(\cdot) = K(\cdot/h)/h$ and $K(\cdot)$ denotes a density function and $h > 0$ a bandwidth. We show in Lemma B.1 that our estimator of $G_d^+(y) = \int_{\mathbb{Y}} \mathbf{1}\{\tilde{y} \leq y\} \max\{0, g_d(\tilde{y})\} d\tilde{y}$ under our assumptions. We therefore define

$$\widehat{G}_d^+(y) = \int_{\mathbb{Y}} \mathbf{1}\{\tilde{y} \leq y\} \max\{0, \widehat{g}_d(\tilde{y})\} d\tilde{y},$$

where $\widehat{g}_d(y) = (\widehat{q}_{dd}(y) - \widehat{q}_{d(1-d)}(y))/\widehat{\pi}_\Delta$. The conditional probability functions are further estimated by $\widehat{Q}_{dz}(y) = \int_{\mathbb{Y}} \mathbf{1}\{\tilde{y} \leq y\} \widehat{q}_{dz}(\tilde{y}) d\tilde{y}$. Based on these estimators, the parameters of θ are estimated and we estimate $\phi_b(\widehat{\theta}_b, \pi_{\text{DF}})$ by simple plug-in methods, where infimum, supremum and integrals are numerically evaluated.

C.5. Assumptions for a Continuous Outcome Variable. We first impose the following regularity assumptions.

Assumption C.2. (i) \mathbb{Y}_d is given by $[\underline{y}_d, \bar{y}_d]$ for $\infty < \underline{y}_d < \bar{y}_d < \infty$ for $d \in \{0, 1\}$. (ii) $\forall d, z \in \{0, 1\}$, the functions $q_{dz}(y)$ are bounded and bounded away from zero, absolutely continuous and two times continuously differentiable with uniformly bounded derivatives. (iii) For $d \in \{0, 1\}$, the functions $q_{dd}(y)$ and $q_{d(1-d)}(y)$ cross at a finite number of times.

Assumption (i) assumes compact support of the outcome variable as it simplifies the following analysis. Assumption (ii) imposes smoothness conditions on the joint densities, which are standard in the nonparametric literature. Assumption (iii) is imposed for simplicity and substantially simplifies the analysis of the estimator of the function $G_d^+(y)$.³⁹

Assumption C.3. (i) The kernel is a second order kernel function, being symmetric around zero, integrates to one, twice continuously differentiable, of bounded variation and zero-valued off, say $[-0.5, 0.5]$. (ii) The bandwidth satisfies: (a) $nh^4 \rightarrow 0$, (b) $nh^2 \rightarrow \infty$, (c) $nh/\log(n) \rightarrow \infty$.

Assumption C.3 (i) imposes conditions on the choice of kernel which can be satisfied by construction and Assumption C.3 (ii) imposes conditions on the bandwidth.

C.6. Asymptotic Results for a Continuous Outcome Variable. We first note that we have the following result.

Proposition C.2. *Suppose Assumptions C.1–C.3 hold. It then follows that*

$$\sqrt{n}(\hat{\theta}(y) - \theta(y)) \rightarrow \mathcal{Z}_1(y),$$

where $\mathcal{Z}_1(y)$ is a tight mean-zero Gaussian process in $\ell^\infty(\mathbb{R}, \mathbb{R}^6)$.⁴⁰

As explained in the main text, we cannot directly base our inference procedure on the mapping $\phi(\theta, \pi_{DF})$, as this mapping is non-smooth and standard asymptotic theory cannot be applied. We, therefore, consider a smoothed version of this mapping in this section. To be more precise, we consider the definition of Masten and Poirier (2020), which we cite here for completeness.

³⁹This assumption is satisfied if the weighted densities $\pi_{DF}f_{Y_d^{DF}}$ and $\pi_{CO}f_{Y_d^{CO}}$ intersect only finitely many times. Without this assumption, our proposed estimator of $\hat{G}_d^+(y)$ is a biased estimator of $G_d^+(y)$. Following the arguments of Anderson, Linton, and Whang (2012), one can construct a debiased estimator of $G_d^+(\bar{y})$, which converges in \sqrt{n} to a mean-zero normal distribution. Based on similar arguments, one could now construct a debiased estimator of $G_d^+(y)$. As this is a rather tedious exercise and not the purpose of this paper, we impose this stronger assumption.

⁴⁰Let A be some arbitrary set and B a Banach space. Then $\ell^\infty(A, B)$ denotes the set of all mappings $f : A \rightarrow B$, which satisfy that $\sup_{a \in A} \|f(a)\|_B \leq \infty$.

Definition C.1 (Definition 1, Masten and Poirier (2020)). Let $(\Theta, \|\cdot\|_\Theta)$ and $(\mathcal{H}, \|\cdot\|_\mathcal{H})$ be Banach spaces. Let \leq be a partial order on \mathcal{H} . Let $h : \Theta \rightarrow \mathcal{H}$ be a function. Consider a function $H_\kappa : \Theta \rightarrow \mathcal{H}$, where $\kappa \in \mathbb{R}_+^{\dim(\kappa)}$ is a vector of smoothing parameters. Then H_κ denotes a *smooth lower approximation* (SLA) of H if

1. Lower envelope: $H_\kappa(\theta) \leq H(\theta)$ for all $\theta \in \Theta$ and $\kappa \in \mathbb{R}_+^{\dim(\kappa)}$.
2. Approximating: For each $\theta \in \Theta$, $H_\kappa(\theta) \rightarrow H(\theta)$ for $\kappa \rightarrow \infty$ (pointwise).
3. Smoothing: H_κ is Hadamard-differentiable.

This definition of a *smooth upper approximation* (SUA) is analogous. We now assume that ϕ_κ is a SLA of ϕ componentwise, and we show in the subsequent sections how we can obtain such a smooth mapping. Let $\hat{\theta}^*$ denotes a draw from the nonparametric bootstrap. We then choose the critical value such that

$$\widehat{cv}_{1-\alpha} = \inf \{z \in \mathbb{R} : \mathbb{P} \left(\left(\sup_{\pi_{\text{DF}} \in [0,0.5], l \leq 5} \sqrt{ne_l}^\top (\phi_\kappa(\hat{\theta}^*, \pi_{\text{DF}}) - \phi_\kappa(\hat{\theta}, \pi_{\text{DF}})) \leq z \mid \{Y_i^z, D_i^z\}_{i=1}^{n_z}\}_{z=0}^1 \right) \geq 1 - \alpha \right\}$$

We can also allow that z is a known function of π_{DF} and l . By doing so, we can exploit the trade-off by constructing the confidence set for the sensitivity and robust region. We construct our function $\phi_{\kappa,L}(\theta, \pi_{\text{DF}}) = \phi_\kappa(\hat{\theta}, \pi_{\text{DF}}) + \widehat{cv}_{1-\alpha}/\sqrt{n}$ and our confidence sets for the sensitivity and robust region $\widehat{\text{SR}}_L(\kappa)$ and $\widehat{\text{RR}}_L(\kappa)$ are constructed based on this mapping as explained in Section 5. We then have the following result.

Proposition C.3. *Suppose Assumptions 1 and Assumptions C.1–C.3 hold. It then follows that $\lim_{n \rightarrow \infty} \mathbb{P}(\widehat{\text{RR}}_L(\kappa) \subseteq \text{RR}, \text{SR} \subseteq \widehat{\text{SR}}_L(\kappa)) \geq 1 - \alpha$.*

C.7. Population Smoothing.

C.7.1. *General Introduction.* We now show how to construct these smoothed mapping ϕ_κ . As our mapping ϕ is a mapping of many non-differentiable mappings, we prove a chain-rule argument, which allows us to consider simpler mappings.

Lemma C.1. *Let ψ and ϕ be two positive and nondecreasing mappings and denote by $\psi^U(\kappa)$ and by $\phi^U(\kappa)$ there respectively SLA, then $\psi^U(f, \kappa)$ and by $\phi^U(\psi^U(f, \kappa), \kappa)$ is a SLA of $\psi(\phi)$. Accordingly, $\phi^U(\psi^U(f, \kappa), \kappa)$ denotes the SUA.*

Based on these definitions, we argue that the mapping $\phi(\theta, \pi_{\text{DF}})$ is a composition of non-smooth random functions, where we replace each of them with a respective SLA and SUA. We first consider

these mapping separately, and we then show how to use them to construct our bounds. Let $\kappa > 1$ be the smoothing parameter. Let $(\Theta, \|\cdot\|_\Theta)$ and $(\mathcal{H}, \|\cdot\|_\mathcal{H})$ be Banach spaces, where \leq is a partial order on \mathcal{H} . we consider two mappings $f, g : \Theta \rightarrow \mathcal{H}$ in the following, which are both Hadamard-differentiable.

Maximum and minimum: We first consider the function $\psi_{av}(f) = |f|$, where a SLA and SUA is given by $\psi_{av}^U(f; \kappa) = \sqrt{f^2 + 1/\kappa}$ and $\psi_{av}^L(f; \kappa) = f^2/(\sqrt{f^2 + 1/\kappa})$.

Lemma C.2. $\psi_{av}^L(f; \kappa)$ is a SLA and $\psi_{av}^U(f; \kappa)$ a SUA for the mapping $\psi_{av}(f)$ and .

Let $\psi_{\min}(f, g) = \min(f, g)$ and $\psi_{\max}(f, g) = \max(f, g)$. A SLA of $\psi_{\max}(f, g)$ is clearly given by $\psi_{\max}^L(f, g; \kappa) = f + g + \psi_{av}^L(f - g; \kappa)$ and a SUA is given by $\psi_{\max}^U(f, g; \kappa) = f + g + \psi_{av}^U(f - g; \kappa)$. It follows from a simple induction argument, that one can generalize this procedure to the maximum of a set of finitely many mappings.

Supremum and infimum: In the following, we consider the mapping $\psi_{\sup, \leq}(f, g)(\cdot) = \sup_{z \leq \cdot} f(z) - g(z)$ and the equally binned set $\mathbb{Y} = \bigcup_{k=1}^{\kappa} [\underline{y} + (k-1)d_Y, \underline{y} + kd_Y]$, where $d_Y = \frac{1}{\kappa}(\bar{y} - \underline{y})$. Let $k_j = \underline{y} + j \cdot d_y$, where $j \in \{0, 1, 2, \dots, \kappa\}$.

$$\begin{aligned}\psi_{\sup, \leq}^L(f, g; \kappa)(\cdot) &= \psi_{\max}^L(\{g(k_j) - f(k_j)\}_{j:k_j \leq \cdot}; \kappa). \\ \psi_{\sup, \leq}^U(f, g; \kappa)(\cdot) &= \psi_{\max}^U(\{g(k_j) - f(\min(\cdot, k_{j+1}))\}_{j:k_j < \cdot}; \kappa).\end{aligned}$$

We similarly define for the mapping $\psi_{\inf, \leq}(f, g)(\cdot) = \inf_{z \leq \cdot} f(z) - g(z)$ that

$$\begin{aligned}\psi_{\inf, \leq}^U(f, g; \kappa)(\cdot) &= \psi_{\min}^U(\{g(k_j) - f(k_{j+1})\}_{j:k_j \leq \cdot}; \kappa). \\ \psi_{\inf, \leq}^L(f, g; \kappa)(\cdot) &= \psi_{\min}^L(\{g(\min(\cdot, k_{j+1})) - f(k_j)\}_{j:k_j < \cdot}; \kappa).\end{aligned}$$

Lemma C.3. If f and g are monotone increasing, $\psi_{\inf, \leq}^L(f, g; \kappa)$ is a SLA and $\psi_{\sup, \leq}^U(f, g; \kappa)$ a SUA to the function $\psi_{\sup, \leq}(f, g)$, and $\psi_{\inf, \leq}^U(f, g; \kappa)$ a SUA and $\psi_{\inf, \leq}^L(f, g; \kappa)$ a SLA to the function $\psi_{\inf, \leq}(f, g)$.⁴¹

C.7.2. *Smoothing the Sensitivity and Robust Regions.* We derive the smoothed mapping

$$\phi_\kappa(\theta, \pi_{DF}) = \left(\underline{\pi}_{DF}^L(\kappa), -\bar{\pi}_{DF}^U(\kappa), \underline{\delta}^L(\pi_{DF}; \kappa), -\bar{\delta}^U(\pi_{DF}; \kappa), BP^L(\pi_{DF}; \kappa) \right).$$

Since our sharp bounds $\underline{F}_{Y_d^{CO}}$ and $\bar{F}_{Y_d^{CO}}$ are the key elements in our construction, we consider them first. We show how to smooth the lower bound from above. we note that $G_d^{\text{sup}}(y) =$

⁴¹By similar reasoning, the functions $\psi_{\sup, \geq}(f, g)$ and $\psi_{\inf, \geq}(f, g)$ can be smoothly approximated.

$\sup_{z \leq y} G_d^+(z) - G_d^-(z)$, where $G_d^-(z) = G_d^+(y) - G_d(y)$. We denote the upper bound by

$$G_d^{\text{sup},U}(y; \kappa) = \psi_{\text{sup},\leq}^U(G_d^+ - G_d^-; \kappa)(y)$$

The bound on the outcome distribution of compliers are therefore bounded by

$$\begin{aligned} \underline{H}_{Y_d^{CO}}^U(y, \pi_{DF}, \delta; \kappa) = \\ \frac{1}{\pi_{CO}} \psi_{\text{max}}^U \left(\left\{ 0, \pi_{\Delta} G_d^{\text{sup},U}(y; \kappa), Q_{\text{dd}}(y) - \pi_d, \frac{\pi_{CO}}{\pi_{\Delta}} (G_d^{\text{sup},U}(y; \kappa) - \pi_{DF} \delta) \right\}; \kappa \right) \end{aligned}$$

A SUA of $\underline{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta)$ is given by

$$\begin{aligned} \underline{F}_{Y_d^{CO}}^U(y, \pi_{DF}, \delta; \kappa) = \frac{1}{\pi_{CO}} Q_{\text{dd}}(y) \\ - \frac{1}{\pi_{CO}} \psi_{\text{inf},\geq}^L \left(Q_{\text{dd}}(\tilde{y}) - \pi_{\Delta} G_d^+(\tilde{y}) - \psi_{\text{inf},\leq}^L \left(\pi_{\Delta} G_d^+(\tilde{y}) - \pi_{CO} \underline{H}^U(y, \pi_{DF}, \delta; \kappa); \kappa \right) (\tilde{y}); \kappa \right) (\tilde{y}). \end{aligned}$$

A SLA of $\underline{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta)$ can be similarly constructed as well as a smooth lower and upper approximation of $\bar{F}_{Y_d^{CO}}(y, \pi_{DF}, \delta)$. We now turn to the sensitivity region. The lower bounds on the sensitivity parameter $\bar{\pi}_{DF}$ can be constructed by

$$\bar{\pi}_{DF}^L(\kappa) = \psi_{\text{min}}^L(\{\mathbb{P}(D = 1|Z = 0), \mathbb{P}(D = 0|Z = 1)\}; \kappa)$$

and similarly upper bound on the sensitivity parameter $\underline{\pi}_{DF}$ by

$$\underline{\pi}_{DF}^U(\kappa) = \frac{\pi_{\Delta}}{\pi_{CO}} \psi_{\text{max}}^U(\{G_1^+(\bar{y}), G_0^+(\bar{y})\}; \kappa) - 1.$$

One can similarly derive the other parameters.

C.8. Illustration of Derivation of Bounds on Outcome Distributions. In Figure C.1, we give some intuition on how the outcome distribution of compliers is constructed. We plot the functions q_{11} and q_{10} . Based on the reasoning of the main text, the function q_{11} is a weighted average of the densities of $f_{Y_1^{CO}}$ and $f_{Y_1^{AT}}$, and the function q_{10} of the densities of $f_{Y_1^{DF}}$ and $f_{Y_1^{AT}}$. It is clear that to be a feasible candidate of $f_{Y_1^{CO}}$, any density has to satisfy that

$$\frac{1}{\pi_{\Delta}} \max\{0, q_{11}(y) - q_{10}(y)\} \leq f_{Y_1^{CO}}(y) \leq \frac{1}{\pi_{CO}} q_{11}(y).$$

In Figure C.1 (a), the density of $f_{Y_1^{CO}}$ is point identified for the sensitivity parameter π_{DF} that is the smallest when ignoring the distribution functions in the absence of treatment. However, if π_{DF} increases the density of $f_{Y_1^{CO}}$ is in general not point identified. The corresponding probability

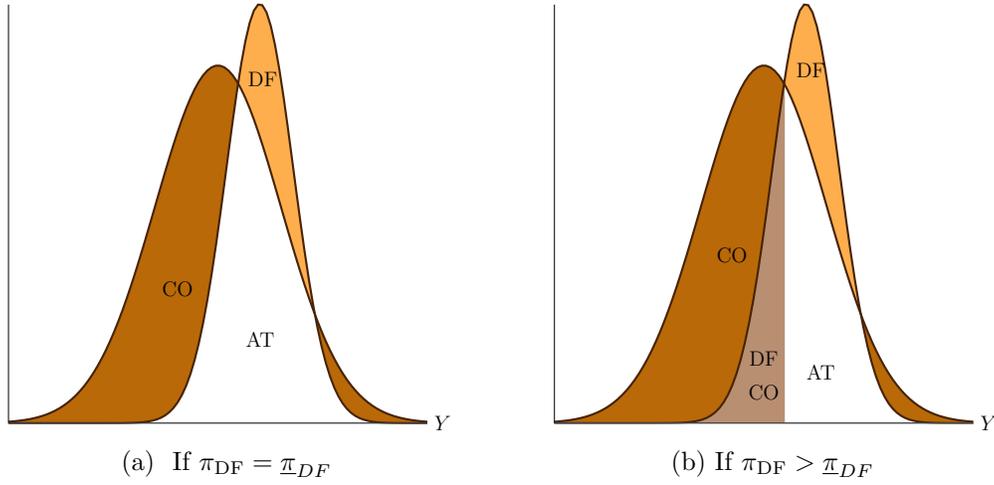


Figure C.1: Derivation of the compliers outcome distributions

mass of the tails of the function $\min\{q_{11}(y), q_{10}(y)\}$ is then imputed to belong to the compliers and defiers. Figure C.1 (b) gives such an example for a possible candidate of density function of $f_{Y_1^{CO}}$ implying an upper bound on the distribution function of compliers.

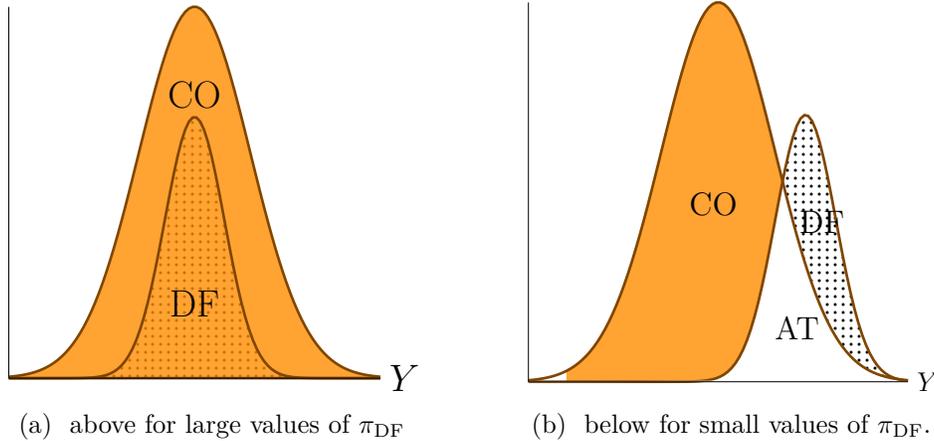


Figure C.2: Illustration of sensitivity region.

C.9. Intuition for Lower and Upper Bound on Outcome Heterogeneity. We give some intuition on how the bounds on the sensitivity parameters δ are derived. Let us first consider the largest value $\bar{\pi}_{DF}$ ignoring the distribution functions in the absence of treatment. In Figure C.2 (a), this value implies that both the outcome distributions of compliers and defiers are point identified, as the population size of always takers would be zero. Thus the outcome distribution function of defiers equals $Q_{10}(y)$, and of compliers equals $Q_{11}(y)$ up to normalization. In this specific example,

the outcome heterogeneity would be point identified but especially bounded from above by 0.5. In Figure C.2 (b), we consider the smallest possible value of outcome heterogeneity $\pi_{DF} = \underline{\pi}_{DF}$. The two outcome distributions are again point identified, and the outcome heterogeneity would be close to one, but especially it would be bounded from below. A similar reasoning then also applies to the absence of treatment.

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